

EFFECTIVENESS OF DEMAILLY'S STRONG OPENNESS CONJECTURE AND RELATED PROBLEMS

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ABSTRACT. In this article, stimulated by the effectiveness in Berndtsson's solution of the openness conjecture and continuing our solution of Demailly's strong openness conjecture, we discuss conditions to guarantee the effectiveness of the conjecture and establish such an effectiveness result. We explicitly point out a lower semicontinuity property of plurisubharmonic functions with a multiplier, which is implicitly contained in [20]. We also obtain optimal effectiveness of the conjectures of Demailly-Kollár and Jonsson-Mustată respectively.

1. INTRODUCTION

1.1. Background: Strong openness conjecture.

In [19, 20], the authors solved the strong openness conjecture posed by Demailly in [6] and [7] (see also [9], [11], [3], [29], [24], [22], [4], [25], [44], [30], etc.):

Strong openness conjecture: *Let φ be a plurisubharmonic function on a complex manifold. Then*

$$\mathcal{I}_+(\varphi) = \mathcal{I}(\varphi),$$

where $\mathcal{I}(\varphi)$ is the multiplier ideal sheaf and $\mathcal{I}_+(\varphi) := \cup_{\varepsilon>0} \mathcal{I}((1+\varepsilon)\varphi)$.

Recall that the multiplier ideal sheaf $\mathcal{I}(\varphi)$ is the sheaf of germs of holomorphic functions f such that $|f|^2 e^{-\varphi}$ is locally integrable (see [32], see also [40], [41], [7], etc.).

For $\dim X \leq 2$, the strong openness conjecture was proved in [24] by studying the asymptotic jumping numbers for graded sequences of ideals.

It is not hard to see that the truth of the strong openness conjecture is equivalent to the following theorem:

Theorem 1.1. [19, 20] *Let φ be a negative plurisubharmonic function on the unit polydisc $\Delta^n \subset \mathbb{C}^n$, which satisfies*

$$\int_{\Delta^n} |F|^2 e^{-\varphi} d\lambda_n < +\infty,$$

where $d\lambda_n$ is the Lebesgue measure on \mathbb{C}^n , F is a holomorphic function on Δ^n . Then there exists a number $p > 1$, such that

$$\int_{\Delta_r^n} |F|^2 e^{-p\varphi} d\lambda_n < +\infty,$$

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where $r \in (0, 1)$.

Assuming $\mathcal{I}(\varphi) = \mathcal{O}_X$, the strong openness conjecture degenerates to the openness conjecture posed by Demailly and Kollár in [10]. The dimension two case of the Openness conjecture was proved by Favre and Jonsson in [13] (see also [12]).

Recently, Berndtsson [2] proved the openness conjecture. Actually, Berndtsson also obtained effectiveness in his solution of the openness conjecture.

Therefore it is natural to ask a similar effectiveness problem in the case of the strong openness conjecture:

Question 1.2. *Under what kinds of conditions, one can find effective $p > 1$, such that*

$$(F, z_0) \in \mathcal{I}(p\varphi)_{z_0}.$$

In the following subsection, we will discuss Question 1.2 and find suitable conditions.

1.2. Effectiveness of the strong openness conjecture.

Let D be a given pseudoconvex domain in \mathbb{C}^n , and z_0 be a point in D . Let φ be a negative plurisubharmonic function on D , and F be a holomorphic function on D .

Denote by

$$\|F\|_{\varphi} := \left(\int_D |F|^2 e^{-\varphi} d\lambda_n \right)^{1/2}.$$

Let C_1 be a positive constant.

First natural condition is:

$$1) \|F\|_{\varphi}^2 \leq C_1.$$

The following example shows that under this condition there doesn't exist an effective (only depending on C_1) number $p > 1$, such that $(F, z_0) \in \mathcal{I}(p\varphi)_{z_0}$:

For $n = 1$ case, let $D = \Delta$, $z_0 = 0$, and $F = z^m$, and $\varphi = 2m \log |z|$. Then p must be smaller than $1 + \frac{1}{m}$. Note that $\int_{\Delta} |F|^2 e^{-\varphi} d\lambda_n = \pi$ for any m . Thus p depends on F .

Thus the condition 1) is not enough to answer Question 1.2.

Then we consider the following modified conditions:

$$1) \|F\|_{\varphi}^2 \leq C_1;$$

2) given F .

The following example shows that under this condition there still doesn't exist an effective number $p > 1$ (only depending on C_1 and F), such that $(F, z_0) \in \mathcal{I}(p\varphi)_{z_0}$:

For $n = 2$ case, let $D = \mathbb{B}^2$, $z_0 = (0, 0)$, $F = z_1$, and $\varphi_{\theta, \delta} = 2\delta \log(|z_1 \cos \theta + z_2 \sin \theta|)$. Note that for any $\delta \in (0, 1)$, there exists $\theta = \theta(\delta) > 0$ small enough, such that

$$\int_{\Delta^2} |F|^2 e^{-\varphi_{\theta, \delta}} d\lambda_n < 2\pi^2.$$

However,

$$\int_{\Delta_r^2} |F|^2 e^{-\frac{1}{\delta} \varphi_{\theta, \delta}} d\lambda_n = +\infty$$

for any $r \in (0, 1)$. When δ go to zero, p must goes to 1.

Thus the modified conditions 1) and 2) are not enough to answer Question 1.2.

Therefore, in order to answer Question 1.2, we need to consider the pair (F, φ) instead of F and φ separately.

We generalize the Bergman kernel K on D to $K_{\varphi, F}$ as follows:

$$K_{\varphi, F}(z_0) := \frac{1}{\inf\{\|F_1\|_0^2 | (F_1 - F, z_0) \in \mathcal{I}_+(2c_{z_0}^F(\varphi)\varphi)_{z_0} \text{ \& } F_1 \in \mathcal{O}(D)\}}$$

where $c_{z_0}^F(\varphi) := \sup\{c \geq 0 : |F|^2 e^{-2c\varphi} \text{ is } L^1 \text{ on a neighborhood of } z_0\}$ is the jumping number (see [25]). Especially, when $F \equiv 1$, $c_{z_0}^F(\varphi)$ will degenerate to the complex singularity exponent $c_{z_0}(\varphi)$ (or log canonical threshold) (see [43, 35, 27, 10], etc.).

It is clear that when $F \equiv 1$ and φ is the pluricomplex Green function $G(z, z_0)$ on D (i.e the upper envelop of negative plurisubharmonic functions on D , which satisfy $G_{z,w} - n \log |z - w|^2$ is locally finite near w , see [5]), then $K_{\varphi, F}(z_0)$ degenerates to the Bergman kernel $K(z_0)$.

Let D be $\Delta \subset \mathbb{C}$. When $F = z^m$ and $\varphi = \log |z|^{2m}$, then

$$K_{\varphi, F}(o) = \left(\int_{\Delta} |z|^{2m} \lambda_1 \right)^{-1} = \frac{m+1}{\pi}.$$

Let D be $\Delta^2 \subset \mathbb{C}^2$. When $F = z_1 \cos \theta + z_2 \sin \theta$ and $\varphi = 2\delta \log |z_1|$, then

$$K_{\varphi, F}(o) = (\sin^2 \theta \int_{\Delta^2} |z_2|^2 \lambda_2)^{-1} = \frac{2}{\pi^2 \sin^2 \theta}.$$

Let D be $\Delta^2 \subset \mathbb{C}^2$. When $F = z_1 + z_2^2$ and $\varphi = 2 \log |z_1|$, then

$$K_{\varphi, F}(o) = \left(\int_{\Delta^2} |z_2|^4 \lambda_2 \right)^{-1} = \frac{4}{\pi^2}.$$

We define a useful function to establish the effectiveness of the strong openness conjecture:

$$\theta(t) := \left(\frac{1}{(t-1)(2t-1)} \right)^{\frac{1}{t}},$$

where $t \in (1, +\infty)$.

In the present article, we establish the effectiveness of the strong openness conjecture as follows:

Theorem 1.3. *Let C_1 and C_2 be two positive constants. We consider the set of the pairs (F, φ) satisfying*

$$1) \ \|F\|_{\varphi}^2 \leq C_1;$$

$$2) \ K_{\varphi, F}^{-1}(z_0) \geq C_2.$$

Then for any $p > 1$ satisfying

$$\theta(p) > \frac{C_1}{C_2},$$

we have

$$(F, z_0) \in \mathcal{I}(p\varphi)_{z_0}.$$

Note that the proof of Theorem 1.3 does not depend on the truth of the strong openness conjecture, then Theorem 1.3 can be regarded as a presentation of our proof of the strong openness conjecture.

Especially, letting $F \equiv 1$ in Theorem 1.3, noting that

$$K_{\varphi,1}(z_0) \leq K(z_0),$$

one can obtain Berndtsson's effectiveness of the openness conjecture in [2]:

Corollary 1.4. *Let C_1 and C_2 be two positive constants. We consider the set of φ satisfying*

- 1) $\|1\|_{\varphi}^2 = \int_D e^{-\varphi} d\lambda_n \leq C_1$;
- 2) $K^{-1}(z_0) \geq C_2$.

Then for any $p > 1$ satisfying

$$\theta(p) > \frac{C_1}{C_2}, \quad (1.1)$$

we have $e^{-p\varphi}$ is integrable near z_0 .

When $D = \mathbb{B}^n$, recall that ε_0 in [2],

$$\varepsilon_0 := \inf\{\|w\|_0^2 \mid \exists z_0 \in \mathbb{B}_{1/2}^n, |w(z_0)| = \frac{1}{10}\} = \inf_{z_0 \in \mathbb{B}_{1/2}^n} \inf\{\|w\|_0^2 \mid |w(z_0)| = \frac{1}{10}\},$$

and

$$K(z_0) = \sup_{w \in \mathcal{O}_{\mathbb{B}^n}} \frac{|w(z_0)|^2}{\|w\|_0^2} = \sup_{|w(z_0)| = \frac{1}{10}} \frac{|w(z_0)|^2}{\|w\|_0^2} = \frac{\frac{1}{100}}{\inf_{|w(z_0)| = \frac{1}{10}} \|w\|_0^2}.$$

It follows that

$$\varepsilon_0 = \inf_{z_0 \in \mathbb{B}_{1/2}^n} \frac{1}{100} \frac{1}{K(z_0)}. \quad (1.2)$$

The following Remark tells us that Corollary 1.4 is (a more precise version of) Berndtsson's effectiveness result of the openness conjecture:

Remark 1.5. *When $D = \mathbb{B}^n$, and $|z_0| \leq \frac{1}{2}$, Berndtsson in [2] pointed out that if p satisfies*

$$\frac{1}{(p-1)} \geq \|1\|_{\varphi}^2 \frac{2}{\varepsilon_0},$$

then $e^{-p\varphi}$ is integrable near z_0 .

One can check that

$$\frac{1}{200(t-1)} < \frac{1}{6(t-1)} < \frac{t}{6(t-1)} < \left(\frac{1}{(t-1)(2t-1)}\right)^{\frac{1}{t}} = \theta(t) \quad (1.3)$$

for any $t \in (1, +\infty)$.

By inequality 1.1, 1.2 and 1.3, it follows that Corollary 1.4 is (a more precise version of) Berndtsson's effectiveness of the openness conjecture.

In subsection 6.1, we show that

$$\frac{1}{\sqrt{3}e^{\frac{2}{t}}(t-1)} < \left(\frac{1}{(t-1)(2t-1)}\right)^{\frac{1}{t}}. \quad (1.4)$$

In subsection 6.2, we give a more precise form of inequality 1.4.

We establish Theorem 1.3 by the following Proposition:

Proposition 1.6. *Let C_1 and C_2 be two positive constants. We consider the set of the triples (F, φ, p) satisfying*

$$1) \|F\|_{D, \varphi}^2 \leq C_1;$$

$$2) C_{F, p\varphi}(z_0) \geq C_2,$$

where $p > 1$ is a real number and $C_{F, p\varphi}(z_0)$ is the infimum of $\int_D |F_1|^2 d\lambda_n$ for all $F_1 \in \mathcal{O}(D)$ satisfying condition $(F_1 - F, z_0) \in \mathcal{I}(p\varphi)_{z_0}$.

Then p must satisfy

$$\theta(p) \leq \frac{C_1}{C_2}. \quad (1.5)$$

1.3. A lower semicontinuity property of plurisubharmonic functions with a multiplier.

In [10], Demailly and Kollár conjectured that:

For every nonzero holomorphic function f on X , there is a number $\delta = \delta(f, K, L) > 0$, such that for any holomorphic function g on X with

$$\sup_L |g - f| < \delta \Rightarrow c_K(\log |g|) \geq c_K(\log |f|),$$

where the compact set K contained in an open subset L of complex manifold X .

In [10], the authors proved that the above conjecture is implied by the ACC conjecture (see [35] or [27]). The ACC conjecture was proved by Hacon, McKernan and Xu in [23].

Note that $c \log |f|$ is a plurisubharmonic function, then we [20] replaced $c \log |f|$ by general plurisubharmonic functions, and obtained the following lower semicontinuity property of plurisubharmonic functions, which is a new proof of the above conjecture in [10] without using the ACC conjecture:

Proposition 1.7. [20] *Let $\{\phi_m\}_{m=1,2,\dots}$ be a sequence of negative plurisubharmonic functions on Δ^n , which is convergent to a negative Lebesgue measurable function ϕ on Δ^n in Lebesgue measure. Assume that $e^{-\phi_m}$ are all not integrable near o . Then $e^{-\phi}$ is not integrable near o .*

In fact, our proof of Proposition 1.7 in [20] already contains the following lower semicontinuity property of plurisubharmonic functions on multiplier ideal sheaves:

Proposition 1.8. *Let $\{\phi_m\}_{m=1,2,\dots}$ be a sequence of negative plurisubharmonic functions on Δ^n , which is convergent to a negative Lebesgue measurable function ϕ on Δ^n in Lebesgue measure. Let $\{F_m\}_{m=1,2,\dots}$ be a sequence of holomorphic functions on Δ^n with uniform bound, which is convergent to a Lebesgue measurable function F on Δ^n in Lebesgue measure. Assume that for any neighborhood U of o , the pairs (F_m, ϕ_m) ($m = 1, 2, \dots$) satisfying*

$$\inf_m K_{\phi_m, F_m}^{-1}(o) > 0,$$

where K_{ϕ_m, F_m} is the generalized Bergman kernel on U . Then $|F|^2 e^{-\varphi}$ is not integrable near o . If ϕ is plurisubharmonic and F is holomorphic, then

$$(F, o) \notin \mathcal{I}(\phi)_o.$$

Especially, letting $F \equiv 1$ in Proposition 1.8, we obtain Proposition 1.7.

In fact, Proposition 1.8 can be regarded as another presentation of our proof of the strong openness conjecture.

Another presentation of Proposition 1.8 is:

Let $\{\phi_m\}_{m=1,2,\dots}$ be a sequence of negative plurisubharmonic functions on Δ^n , which is convergent to a plurisubharmonic function ϕ on Δ^n in Lebesgue measure.

Let $\{F_m\}_{m=1,2,\dots}$ be a sequence of holomorphic functions on Δ^n with uniform bound, which is convergent to a holomorphic function F on Δ^n in Lebesgue measure.

Assume that for any neighborhood U of o , the pairs (F_m, ϕ_m) ($m = 1, 2, \dots$) satisfying

$$\inf_m K_{\phi_m, F_m}^{-1}(o) > 0,$$

There exists m_0 , such that for any $m \geq m_0$,

$$c_o^{F_m}(\varphi_m) \geq c_o^F(\varphi),$$

where $c_o^F(\psi) = \sup\{c \geq 0 : |F|^2 e^{-2c\psi}$ is L^1 on a neighborhood of $o\}$ is the jumping number (see [25]).

1.4. Optimal effectiveness of a conjecture posed by Demailly and Kollár.

In [20], we solved the following conjecture about the volume growth of the sublevel sets of plurisubharmonic functions related to the complex singularity exponents posed by Demailly and Kollár in [10] (see also [13], [12], [24] and [25], etc.):

Conjecture D-K: *Let φ be a plurisubharmonic function on $\Delta^n \subset \mathbb{C}^n$, and K be compact subset of Δ^n . If $c_K(\varphi) < +\infty$, then*

$$\frac{1}{r^{2c_K(\varphi)}} \mu(\{\varphi < \log r\})$$

has a uniform positive lower bound independent of $r \in (0, 1)$ small enough, where $c_K(\varphi) = \sup\{c \geq 0 : e^{-2c\varphi}$ is L^1 on a neighborhood of $K\}$, and μ is the Lebesgue volumes on \mathbb{C}^n .

For $n \leq 2$, the above conjecture was proved by Favre and Jonsson in [13] (see also [12]).

In [20], in order to prove Conjecture D-K, we obtained an estimate about the volume growth of the sublevel sets of plurisubharmonic functions:

Theorem 1.9. [20] *Let φ be a plurisubharmonic function on $\Delta^n \subset \mathbb{C}^n$. Let F be a holomorphic function on Δ^n . Assume that $|F|^2 e^{-\varphi}$ is not locally integrable near o . Then*

$$\int_{\Delta^n} \mathbb{I}_{\{-(R+1) < \varphi < -R\}} |F|^2 e^{-\varphi} d\lambda_n$$

has a uniform positive lower bound independent of $R \gg 0$. Especially, if $F = 1$, then

$$e^R \mu(\{-(R+1) < \varphi < -R\})$$

has a uniform positive lower bound independent of $R \gg 0$.

Theorem 1.9 tells us that

$$\liminf_{R \rightarrow +\infty} \int_{\Delta^n} \mathbb{I}_{\{-(R+1) < \varphi < -R\}} |F|^2 e^{-\varphi} d\lambda_n > 0.$$

In fact, our proof of Theorem 1.9 in [20] already contains the following effectiveness of the uniform positive lower bound in Theorem 1.3:

Proposition 1.10. *Let $B_0 \in (0, 1]$ be arbitrarily given. Let φ be a negative plurisubharmonic function on pseudoconvex domain $D \subset \mathbb{C}^n$. Let F be a holomorphic function on D . Assume that $|F|^2 e^{-\varphi}$ is not locally integrable near z_0 . Then we obtain that*

$$\liminf_{R \rightarrow +\infty} \frac{1}{B_0} \int_D \mathbb{I}_{\{-(R+B_0) < \varphi < -R\}} |F|^2 e^{t_0+B_0} d\lambda_n \geq K_{\varphi, F}^{-1}(z_0)$$

Especially, if $F = 1$, then

$$\liminf_{R \rightarrow +\infty} e^{R+B_0} \frac{1}{B_0} \mu(\{-(R+B_0) < \varphi < -R\}) \geq K_{\varphi, 1}^{-1}(z_0) \geq K^{-1}(z_0). \quad (1.6)$$

Taking $R = kB_0$ in inequality 1.6, for any given $\varepsilon > 0$, there exists k_0 depending on B_0 , such that for any $k \geq k_0$, one can obtain

$$e^{(k+1)B_0} \frac{1}{B_0} \mu(\{-(k+1)B_0 < \varphi < -kB_0\}) \geq (K^{-1}(z_0) - \varepsilon),$$

i.e.,

$$\mu(\{-(k+1)B_0 < \varphi < -kB_0\}) \geq e^{-(k+1)B_0} B_0 (K^{-1}(z_0) - \varepsilon). \quad (1.7)$$

Taking sum with $k \geq k_0$ in inequality 1.7 and letting B_0 goes to 0, one can obtain

$$\liminf_{R \rightarrow +\infty} \mu(\{\varphi < -R\}) e^R \geq (K^{-1}(z_0) - \varepsilon).$$

Then we obtain an optimal estimate of the lower bound of $\liminf_{R \rightarrow +\infty} e^R \mu(\{\varphi < -R\})$

$$\liminf_{R \rightarrow +\infty} e^R \mu(\{\varphi < -R\}) \geq K_{\varphi, 1}^{-1}(z_0) \geq K^{-1}(z_0). \quad (1.8)$$

When $D = \Delta \subset \mathbb{C}$, $\varphi = \log |z|^2$ and $z_0 = 0$, the equality in inequality 1.6 holds.

Replacing R by $-2c_K(\varphi) \log r$ and φ by $2c_K(\varphi)\varphi$, we obtain the optimal effectiveness of Conjecture D-K:

$$\liminf_{r \rightarrow 0} \frac{1}{r^{2c_K(\varphi)}} \mu(\{\varphi < \log r\}) \geq K^{-1}(z_0) \geq \inf_{z \in K} K^{-1}(z),$$

where z_0 satisfies $c_{z_0}(\varphi) = c_K(\varphi)$.

When $D = \Delta \subset \mathbb{C}$, $\varphi = \log |z|^2$ and $K = \{0\}$, the equality in the above inequality holds.

1.5. Optimal effectiveness of a conjecture posed by Jonsson and Mustatǎ.

In [20], we solved the following conjecture about the volumes growth of the sublevel sets of quasi-plurisubharmonic functions posed by Jonsson and Mustatǎ in [25] (see also [24]):

Conjecture J-M: *Let ψ be a plurisubharmonic function on $\Delta^n \subset \mathbb{C}^n$. If $c_o^I(\psi) < +\infty$, then*

$$\frac{1}{r^2} \mu(\{c_o^I(\psi)\psi - \log |I| < \log r\})$$

has a uniform positive lower bound independent of $r \in (0, 1)$ small enough, where I is an ideal of $\mathcal{O}_{\Delta^n, o}$, which is generated by $\{f_j\}_{j=1, \dots, l}$,

$$\log |I| := \log \max_{1 \leq j \leq l} |f_j|,$$

$c_o^I(\psi) = \sup\{c \geq 0 : |I|^2 e^{-2c\psi}$ is L^1 on a neighborhood of o \} is the jumping number in [25].

For $n \leq 2$, the above conjecture was proved by Jonsson and Mustatǎ in [24].

In [20], in order to prove conjecture J-M, we gave the following estimate about the volume growth of the sublevel sets of quasisubharmonic functions:

Theorem 1.11. [20] *Let ψ be a plurisubharmonic function on Δ^n , and F be a holomorphic function on Δ^n . Assume that $|F|^2 e^{-\psi}$ is not locally integrable near o . Then*

$$e^R \frac{1}{B_0} \mu(\{-R - B_0 < \psi - \log |F|^2 < -R\})$$

has a uniformly positive lower bound independent of $R \gg 0$ and $B_0 \in (0, 1]$.

Theorem 1.11 tells us that

$$\liminf_{R \rightarrow +\infty} e^R \frac{1}{B_0} \mu(\{-R - B_0 < \psi - \log |F|^2 < -R\}) > 0$$

In fact, our proof of Theorem 1.11 in [20] already contains the following effectiveness of the uniform positive lower bound in Theorem 1.3:

Proposition 1.12. *Let δ be an arbitrarily given positive integer. Let ψ be a bounded from above plurisubharmonic function on pseudoconvex domain $D \subset \mathbb{C}^n$. Let F be a bounded holomorphic function on D . Assume that $|F|^2 e^{-\psi}$ is not locally integrable near z_0 . Then we obtain that*

$$\liminf_{R \rightarrow +\infty} e^R \frac{1}{B_0} \mu(\{-R - B_0 < \psi - \log |F|^2 < -R\}) \geq \frac{C_{\psi, F, \delta}}{(1 + \frac{1}{\delta}) e^{B_0}}. \quad (1.9)$$

where $C_{\psi, F, \delta} = \frac{K^{-1}}{\sup_D e^{(1+\delta) \max\{\psi, 2 \log |F|\}}}$, and $K_{\psi + \delta \max\{\psi, \log |F|^2\}, F^{1+\delta}}$ is the generalized Bergman kernel on D .

Taking $R = kB_0$ in inequality 1.9, for any given $\varepsilon > 0$, there exists k_0 depending on B_0 , such that for any $k \geq k_0$, one can obtain

$$e^{(k+1)B_0} \frac{1}{B_0} \mu(\{-(k+1)B_0 < \psi - \log |F|^2 < -kB_0\}) \geq (\frac{C_{\psi, F, \delta}}{(1 + \frac{1}{\delta}) e^{B_0}} - \varepsilon),$$

i.e.

$$\mu(\{-(k+1)B_0 < \psi - \log |F|^2 < -kB_0\}) \geq e^{-(k+1)B_0} B_0 (\frac{C_{\psi, F, \delta}}{(1 + \frac{1}{\delta}) e^{B_0}} - \varepsilon). \quad (1.10)$$

Taking sum $k \geq k_0$ in inequality 1.10, and letting B_0 go to zero, one can obtain the following estimate:

$$\liminf_{R \rightarrow +\infty} e^R \mu(\{\psi - \log |F|^2 < -R\}) \geq \sup_{\delta \in \{1, 2, \dots\}} \frac{C_{\psi, F, \delta}}{(1 + \frac{1}{\delta})}. \quad (1.11)$$

By Theorem 1.1, it follows that $|I|^2 e^{-2c_o^I(\psi)\psi}$ is not integrable on any neighborhood of o .

Replacing ψ by $2c_o^I(\psi)\psi$, and R by $-2 \log r$ in equality 1.11, we obtain the optimal effectiveness of conjecture J-M:

$$\begin{aligned} & \liminf_{r \rightarrow 0} \frac{1}{r^2} \mu(\{c_o^I(\psi)\psi - \log |I| < \log r\}) \\ & \geq \sup_{\delta \in \{1, 2, \dots\}} \frac{\max_{1 \leq i \leq l} \{K_{2c_o^I(\psi)\psi + \delta \max\{2c_o^I(\psi)\psi, \log |f_i|^2\}, f_i^{1+\delta}}(o)\}}{(1 + \frac{1}{\delta}) \sup_D e^{(1+\delta) \max\{2c_o^I(\psi)\psi, 2 \log |I|\}}}, \end{aligned} \quad (1.12)$$

where D is a relatively compact pseudoconvex domain in Δ^n , and

$$K_{2c_o^I(\psi)\psi+\delta\max\{2c_o^I(\psi)\psi,\log|f_i|^2\},f_i^{(1+\delta)}}$$

is the generalized Bergman Kernel on D .

Remark 1.13. When $D = \Delta$, $F = 1$, $\psi = \log|z|^2$, and δ goes to ∞ , the equality in inequality 1.12 holds.

Remark 1.14. The optimal effectiveness of conjecture J-M is a generalization of the optimal effectiveness of conjecture D-K.

2. PROOF OF EFFECTIVENESS OF STRONG OPENNESS CONJECTURE

2.1. A Lemma used to prove Proposition 1.6.

We prove Proposition 1.6 by the following Lemma, whose various forms already appear in [16, 18, 20] etc. and whose proof will appear in the section 4 for the sake of completeness:

Lemma 2.1. Let $B_0 \in (0, 1]$ be arbitrarily given. Let D_v be a strongly pseudoconvex domain relatively compact in Δ^n containing o . Let F be a holomorphic function on Δ^n . Let ψ be a negative plurisubharmonic function on Δ^n , such that $\psi(o) = -\infty$. Then there exists a holomorphic function F_{v,t_0} on D_v , such that,

$$(F_{v,t_0} - F, o) \in \mathcal{I}(\psi)_o$$

and

$$\begin{aligned} & \int_{D_v} |F_{v,t_0} - (1 - b_{t_0}(\psi))F|^2 d\lambda_n \\ & \leq (1 - e^{-(t_0+B_0)}) \int_{D_v} \frac{1}{B_0} (\mathbb{I}_{\{-t_0-B_0 < t < -t_0\}} \circ \psi) |F|^2 e^{-\psi} d\lambda_n, \end{aligned} \quad (2.1)$$

where $b_{t_0}(t) = \int_{-\infty}^t \frac{1}{B_0} \mathbb{I}_{\{-t_0-B_0 < s < -t_0\}} ds$, and t_0 is a positive number.

Remark 2.2. Replacing the strong pseudoconvexity of D_v by pseudoconvexity, and o by $z_0 \in D_v$ for any v , Lemma 2.1 also holds.

2.2. Proof of Proposition 1.6.

As $C_{F,p\varphi}(z_0) \geq C_2 > 0$, then we obtain $\varphi(z_0) = -\infty$.

Let $\psi := p\varphi$.

Using Lemma 2.1 and Remark 2.2, we obtain that

$$\begin{aligned} & ((\int_{D_v} |F_{v,t_0}|^2 d\lambda_n)^{1/2} - (\int_{D_v} |(1 - b_{t_0}(\psi))F|^2 d\lambda_n)^{1/2})^2 \\ & \leq \int_{D_v} |F_{v,t_0} - (1 - b_{t_0}(\psi))F|^2 d\lambda_n \\ & \leq (1 - e^{-(t_0+B_0)}) \int_{D_v} \frac{1}{B_0} (\mathbb{I}_{\{-t_0-B_0 < t < -t_0\}} \circ \psi) |F|^2 e^{-\psi} d\lambda_n. \end{aligned} \quad (2.2)$$

Note that

$$\begin{aligned} & (1 - e^{-(t_0+B_0)}) \int_{D_v} \frac{1}{B_0} (\mathbb{I}_{\{-t_0-B_0 < t < -t_0\}} \circ \psi) |F|^2 e^{-\psi} d\lambda_n \\ & \leq (e^{t_0+B_0} - 1) \int_{D_v} \frac{1}{B_0} (\mathbb{I}_{\{-t_0-B_0 < t < -t_0\}} \circ \psi) |F|^2 d\lambda_n, \end{aligned} \quad (2.3)$$

Then we have

$$\begin{aligned} & ((\int_{D_v} |F_{v,t_0}|^2 d\lambda_n)^{1/2} - (\int_{D_v} |(1 - b_{t_0}(\psi))F|^2 d\lambda_n)^{1/2})^2 \\ & \leq (e^{t_0+B_0} - 1) \int_{D_v} \frac{1}{B_0} (\mathbb{I}_{\{-t_0-B_0 < t < -t_0\}} \circ \psi) |F|^2 d\lambda_n, \end{aligned} \quad (2.4)$$

Note that

$$e^{\frac{1}{p}t_0} \int_{D_v} |(1 - b_{t_0}(\psi))F|^2 d\lambda_n \leq e^{\frac{1}{p}t_0} \int_{D_v} \mathbb{I}_{\{\frac{1}{p}\psi \leq -\frac{1}{p}t_0\}} |F|^2 d\lambda_n \leq \int_{D_v} |F|^2 e^{-\frac{1}{p}\psi} d\lambda_n$$

As

$$C_1 \geq \int_{D_v} |F|^2 e^{-\frac{1}{p}\psi} d\lambda_n = \int_{D_v} |F|^2 e^{-\varphi} d\lambda_n.$$

and $\inf\{\int_{D_v} |F_1|^2 d\lambda_n | F_1 \in \mathcal{O}_{D_v}, (F_1 - F, z_0) \in \mathcal{I}(\psi)_{z_0}\} \geq C_2$, when

$$e^{\frac{1}{p}t_0} \geq \frac{C_1}{C_2},$$

then we have

$$\begin{aligned} & (C_2^{1/2} - (C_1 e^{-\frac{1}{p}t_0})^{1/2})^2 \\ & \leq ((\int_{D_v} |F_{v,t_0}|^2 d\lambda_n)^{1/2} - (\int_{D_v} |(1 - b_{t_0}(\psi))F|^2 d\lambda_n)^{1/2})^2. \end{aligned} \quad (2.5)$$

It follows that

$$\begin{aligned} & (C_2^{1/2} - (C_1 e^{-\frac{1}{p}t_0})^{1/2})^2 \\ & \leq (e^{t_0+B_0} - 1) \int_{D_v} \frac{1}{B_0} (\mathbb{I}_{\{-t_0-B_0 < t < -t_0\}} \circ \psi) |F|^2 d\lambda_n \\ & \leq e^{t_0+B_0} \int_{D_v} \frac{1}{B_0} (\mathbb{I}_{\{-t_0-B_0 < t < -t_0\}} \circ \psi) |F|^2 d\lambda_n. \end{aligned} \quad (2.6)$$

Replacing t_0 by kB_0 , and assuming that $e^{\frac{1}{p}kB_0} \geq \frac{C_1}{C_2}$, we obtain that

$$\begin{aligned} & (C_2^{1/2} - (C_1 e^{-\frac{1}{p}kB_0})^{1/2})^2 \\ & \leq e^{(k+1)B_0} \int_{D_v} \frac{1}{B_0} (\mathbb{I}_{\{-(k+1)B_0 < t < -kB_0\}} \circ \psi) |F|^2 d\lambda_n. \end{aligned} \quad (2.7)$$

It follows that

$$\begin{aligned} & B_0 e^{-(k+1)B_0} e^{\frac{1}{p}kB_0} (C_2^{1/2} - (C_1 e^{-\frac{1}{p}kB_0})^{1/2})^2 \\ & \leq e^{\frac{1}{p}kB_0} \int_{D_v} (\mathbb{I}_{\{-(k+1)B_0 < t < -kB_0\}} \circ \psi) |F|^2 d\lambda_n. \end{aligned} \quad (2.8)$$

Taking k_0 , such that

$$e^{\frac{1}{p}k_0 B_0} \geq \frac{C_1}{C_2} \geq e^{\frac{1}{p}(k_0-1)B_0},$$

and taking sum, we obtain

$$\begin{aligned}
& \sum_{k=k_0}^{+\infty} B_0 e^{-(k+1)B_0} e^{\frac{1}{p}kB_0} (C_2^{1/2} - (C_1 e^{-\frac{1}{p}kB_0})^{1/2})^2 \\
& \leq \sum_{k=k_0}^{+\infty} e^{\frac{1}{p}kB_0} \int_{D_v} (\mathbb{I}_{\{-(k+1)B_0 < t < -kB_0\}} \circ \psi) |F|^2 d\lambda_n \leq \int_{D_v} |F|^2 e^{-\frac{1}{p}\psi} d\lambda_n \leq C_1.
\end{aligned} \tag{2.9}$$

Note that

$$\begin{aligned}
& \sum_{k=k_0}^{+\infty} B_0 e^{-(k+1)B_0} e^{\frac{1}{p}kB_0} (C_2^{1/2} - (C_1 e^{-\frac{1}{p}kB_0})^{1/2})^2 \\
& = \sum_{k=k_0}^{+\infty} (B_0 e^{-(k+1)B_0} e^{\frac{1}{p}kB_0} C_2 - 2B_0 e^{-B_0} e^{-(\frac{1}{2p}+1)kB_0} e^{\frac{1}{p}kB_0} C_2^{1/2} C_1^{1/2} \\
& \quad + B_0 e^{-B_0} e^{-(\frac{1}{p}+1)kB_0} e^{\frac{1}{p}kB_0} C_1) \\
& = \frac{B_0}{1 - e^{-(1-\frac{1}{p})B_0}} e^{-(k_0(1-\frac{1}{p})+1)B_0} C_2 - 2 \frac{B_0}{1 - e^{-(1-\frac{1}{2p})B_0}} e^{-(k_0)(1-\frac{1}{2p})B_0-B_0} C_2^{1/2} C_1^{1/2} + \\
& \quad + \frac{B_0}{1 - e^{-B_0}} e^{-(k_0)B_0-B_0} C_2 \\
& \geq \frac{B_0}{1 - e^{-(1-\frac{1}{p})B_0}} \left(\frac{C_2}{C_1}\right)^{p(1-\frac{1}{p})} e^{-B_0-(1-\frac{1}{p})B_0} C_2 \\
& \quad - 2 \frac{B_0}{1 - e^{-(1-\frac{1}{2p})B_0}} \left(\frac{C_2}{C_1}\right)^{p(1-\frac{1}{2p})} e^{-B_0} C_2^{1/2} C_1^{1/2} \\
& \quad + \frac{B_0}{1 - e^{-B_0}} \left(\frac{C_2}{C_1}\right)^p e^{-B_0-B_0} C_1,
\end{aligned} \tag{2.10}$$

Take limitation

$$\begin{aligned}
& \lim_{B_0 \rightarrow 0} \left(\frac{B_0}{1 - e^{-(1-\frac{1}{p})B_0}} \left(\frac{C_2}{C_1}\right)^{p(1-\frac{1}{p})} e^{-B_0-(1-\frac{1}{p})B_0} C_2 \right. \\
& \quad - 2 \frac{B_0}{1 - e^{-(1-\frac{1}{2p})B_0}} \left(\frac{C_2}{C_1}\right)^{p(1-\frac{1}{2p})} e^{-B_0} C_2^{1/2} C_1^{1/2} \\
& \quad \left. + \frac{B_0}{1 - e^{-B_0}} \left(\frac{C_2}{C_1}\right)^p e^{-B_0-B_0} C_1 \right) \\
& = (1 - \frac{1}{p})^{-1} \left(\frac{C_2}{C_1}\right)^{p(1-\frac{1}{p})} C_2 - 2(1 - \frac{1}{2p})^{-1} \left(\frac{C_2}{C_1}\right)^{p(1-\frac{1}{2p})} C_2^{1/2} C_1^{1/2} + \left(\frac{C_2}{C_1}\right)^p C_1 \\
& = (1 - \frac{1}{p})^{-1} \left(\frac{C_2}{C_1}\right)^{p-1} C_2 - 2(1 - \frac{1}{2p})^{-1} \left(\frac{C_2}{C_1}\right)^{p-\frac{1}{2}} C_2^{1/2} C_1^{1/2} + \left(\frac{C_2}{C_1}\right)^p C_1 \\
& = (1 - \frac{1}{p})^{-1} \left(\frac{C_2}{C_1}\right)^p C_1 - 2(1 - \frac{1}{2p})^{-1} \left(\frac{C_2}{C_1}\right)^p C_1 + \left(\frac{C_2}{C_1}\right)^p C_1 \\
& = ((1 - \frac{1}{p})^{-1} - 2(1 - \frac{1}{2p})^{-1} + 1) \left(\frac{C_2}{C_1}\right)^p C_1.
\end{aligned} \tag{2.11}$$

By inequality 2.9, 2.10 and 2.11, it follows that

$$((1 - \frac{1}{p})^{-1} - 2(1 - \frac{1}{2p})^{-1} + 1)(\frac{C_2}{C_1})^p C_1 \leq C_1, \quad (2.12)$$

that is to say

$$(\frac{1}{(p-1)(2p-1)})^{\frac{1}{p}} \leq \frac{C_1}{C_2}, \quad (2.13)$$

Thus we obtain Proposition 1.6.

2.3. Proof of Theorem 1.3.

Note that $C_{F,p\varphi}(z_0) > 0$. It follows that $(F, z_0) \notin \mathcal{I}(p\varphi)_{z_0}$, i.e. $|F|^2 e^{-p\varphi}$ is not integrable near z_0 . Note that

$$C_{F,p\varphi}(z_0) \geq \inf\{\|F_1\|_0^2 | (F_1 - F, z_0) \in \mathcal{I}_+(2c_{z_0}^F(\varphi)\varphi)_{z_0} \& F_1 \in \mathcal{O}(D)\} > 0. \quad (2.14)$$

Using Proposition 1.6, we obtain Theorem 1.3.

3. PROOF OF THE LOWER SEMICONTINUITY PROPERTY OF PLURISUBHARMONIC FUNCTIONS WITH A MULTIPLIER

In this section, we explicitly point out that the proof of Proposition 1.8 is implicitly contained in our proof of Proposition 1.6 in [20].

For the sake of completeness, we recall our proof of Proposition 1.6 in [20] with the following slightly modifications:

- 1) changing μ into μ_F , and $\mu(\Omega)$ into C_0 respectively;
- 2) changing $e^{-\phi}$ into $|F|^2 e^{-\phi}$,

where $C_0 := \inf_m \inf\{\|F_1\|_0^2 | (F_1 - F, z_0) \in \mathcal{I}(\phi_m)_{z_0} \& F_1 \in \mathcal{O}(\Omega)\} > 0$, and $\mu_F := |F|^2 \mu$ on Δ^n (the choice of Ω see the following part of the present section).

Let's recall our proof of Proposition 1.6 in [20] in details:

We prove Proposition 1.8 by contradiction. If $|F|^2 e^{-\phi}$ is integrable near $o \in \Delta^n$, then there exists a strong pseudoconvex domain $\Omega \subset \subset \Delta^n$, such that $|F|^2 e^{-\phi}$ is L^1 integrable on Ω .

Without losing of generality, we assume that $\Omega = \mathbb{B}(o, r)$, where $r > 0$ small enough.

As $|F|^2 e^{-\phi}$ is L^1 integrable on Ω , then

$$\lim_{R \rightarrow +\infty} e^R \mu_F(\{\phi < -R\}) = 0. \quad (3.1)$$

Therefore there exists $t_1 > 0$, such that

$$\mu_F(\{\phi < -t_1 + 1\}) < \frac{1}{6} C_0. \quad (3.2)$$

As $\{\phi_m\}_{m=1,2,\dots}$ is convergent to ϕ , and $\{F_m\}_{m=1,2,\dots}$ with uniform bound, it follows that there exists $m_0 > 0$, such that for any $m \geq m_0$,

$$\mu_{F_m}(\{|\phi_m - \phi| \geq 1\}) < \frac{1}{12} C_0. \quad (3.3)$$

Note that

$$(\{\phi_m < -t_1\} \setminus \{|\phi_m - \phi| \geq 1\}) \subset \{\phi < -t_1 + 1\},$$

for any $m \geq m_0$. Therefore

$$\mu_{F_m}(\{\phi_m < -t_1\}) \leq \mu_{F_m}(\{\phi < -t_1 + 1\}) + \mu_{F_m}(\{|\phi_m - \phi| \geq 1\}) < \frac{1}{4} C_0, \quad (3.4)$$

for any $m \geq m_0$.

In Lemma 2.1 with $B_0 = 1$, then there exists a holomorphic function F_{v,t_0} on Ω , satisfying:

$$(F_{m,v,t_0} - F_m, o) \in \mathcal{I}(\phi_m)_o \quad (3.5)$$

and

$$\begin{aligned} & \int_{\Omega} |F_{m,v,t_0} - (1 - b_{t_0}(\phi_m))F|^2 d\lambda_n \\ & \leq \int_{\Omega} (\mathbb{I}_{\{-t_0-1 < t < -t_0\}} \circ \phi_m) |F_m|^2 e^{-\phi_m} d\lambda_n. \end{aligned} \quad (3.6)$$

It follows from 3.5 that

$$\int_{\Omega} |F_{m,v,t_0}|^2 d\lambda_n \geq C_0. \quad (3.7)$$

It follows from inequality 3.4, equality 3.5, and $b_{t_0}(t)|_{\{t \geq -t_0\}} = 1$, that

$$\begin{aligned} \int_{\Omega} |(1 - b_{t_0}(\phi_m))F_m|^2 d\lambda_n &= \int_{\{\phi < -t_0\} \cap \Omega} |(1 - b_{t_0}(\phi_m))F_m|^2 d\lambda_n \\ &\leq \int_{\{\phi < -t_0\} \cap \Omega} |F_m|^2 d\lambda_n \\ &\leq \mu_{F_m}(\{\phi < -t_0\}) < \frac{1}{4}C_0 \end{aligned} \quad (3.8)$$

It follows from inequalities 3.8 and 3.7 that

$$\begin{aligned} & \left(\int_{\Omega} |F_{m,v,t_0} - (1 - b_{t_0}(\phi_m))F_m|^2 d\lambda_n \right)^{1/2} \\ & \geq \left(\int_{\Omega} |F_{m,v,t_0}|^2 d\lambda_n \right)^{1/2} - \left(\int_{\Omega} |(1 - b_{t_0}(\phi_m))F_m|^2 d\lambda_n \right)^{1/2} \\ & \geq C_0^{1/2} - 2^{-1}C_0^{1/2} = 2^{-1}C_0^{1/2}, \end{aligned} \quad (3.9)$$

for any $t_0 > t_1$ and any $m \geq m_0$.

It follows from inequalities 3.6 and 3.9 that

$$\int_{\Omega} (\mathbb{I}_{\{-t_0-1 < t < -t_0\}} \circ \phi_m) |F_m|^2 e^{-\phi_m} d\lambda_n \geq 2^{-2}C_0,$$

for any $t_0 > t_1$ and any $m \geq m_0$.

Note that

$$\mu_{F_m}(\{-t_0 - 1 < \phi_m < -t_0\}) e^{t_0+1} \geq \int_{\Omega} (\mathbb{I}_{\{-t_0-1 < t < -t_0\}} \circ \phi_m) |F_m|^2 e^{-\phi_m} d\lambda_n,$$

for any $t_0 > t_1$ and any $m \geq m_0$. Therefore

$$\mu_{F_m}(\{-t_0 - 1 < \phi_m < -t_0\}) \geq e^{-t_0-1} 2^{-2}C_0,$$

for any $t_0 > t_1$ and any $m \geq m_0$.

As $\{\phi_m\}_{m=1,2,\dots}$ is convergent to ϕ in Lebesgue measure, and $\{F_m\}_{m=1,2,\dots}$ is uniformly bounded, then there exists large enough positive integer $m_1 \geq m_0$, such that

$$\mu_{F_{m_1}}(\{|\phi_{m_1} - \phi| \geq 1\}) < \frac{1}{2} e^{-t_0-1} 2^{-2}C_0,$$

for any $t_0 > t_1$.

Note that

$$\{\phi < -t_0 + 1\} \supset (\{-t_0 - 1 < \phi_{m_1} < -t_0\} \setminus \{|\phi_{m_1} - \phi| \geq 1\}).$$

Then we have

$$\begin{aligned} \mu_{F_{m_1}}(\{\phi < -t_0 + 1\}) &\geq \mu_{F_{m_1}}(\{-t_0 - 1 < \phi_{m_1} < -t_0\} \setminus \{|\phi_{m_1} - \phi| \geq 1\}) \\ &\geq \mu_{F_{m_1}}(\{-t_0 - 1 < \phi_{m_1} < -t_0\}) - \mu_{F_{m_1}}(\{|\phi_{m_1} - \phi| \geq 1\}) \\ &\geq \frac{1}{2}e^{-t_0-1}2^{-2}C_0, \end{aligned} \tag{3.10}$$

for any $t_0 > t_1$, i.e.

$$e^{t_0-1}\mu_{F_{m_1}}(\{\phi < -t_0 + 1\}) \geq \frac{1}{2}e^{-2}2^{-2}C_0, \tag{3.11}$$

for any $t_0 > t_1$.

When m_1 goes to infinity, as $\{F_m\}_{m=1,2,\dots}$ is convergent to F in Lebesgue measure with uniform bound, by the dominated convergence theorem, it follows that

$$\lim_{m_1 \rightarrow +\infty} e^{t_0-1}\mu_{F_{m_1}}(\{\phi < -t_0 + 1\}) = e^{t_0-1}\mu_F(\{\phi < -t_0 + 1\}).$$

Using inequality 3.11, we obtain

$$e^{t_0-1}\mu_F(\{\phi < -t_0 + 1\}) \geq \frac{1}{2}e^{-2}2^{-2}C_0,$$

which contradicts to equality 3.1.

Proposition 1.8 has thus been proved.

Remark 3.1. *Using the same method as in the above proof with more subtle bounds in inequalities 3.2 and 3.3, one can obtain inequality 3.11 with a lower bound $e^{-2}C_0$.*

4. PROOFS OF THE EFFECTIVENESS OF CONJECTURE D-K AND CONJECTURE J-M

In this section, we explicitly point out that Proposition 1.10 and Proposition 1.12 are already implicitly contained in [20].

4.1. Proof of Proposition 1.10.

In this subsection, we explicitly point out that the proof of Proposition 1.10 is implicitly contained in our proof of Theorem 1.9 in [20].

For the sake of completeness, we recall some steps of our proof of Theorem 1.9 in [20] with slightly modification: changing o into z_0 .

By Proposition 2.1, it follows that there exists F_{v,t_0} , which is a holomorphic function on D_v satisfying:

$$\begin{aligned} &\int_{D_v} |F_{v,t_0} - (1 - b_{t_0}(\varphi))F|^2 d\lambda_n \\ &\leq \int_{D_v} \mathbb{I}_{\{-t_0 - B_0 < \varphi < -t_0\}} |F|^2 e^{-\varphi} d\lambda_n. \end{aligned} \tag{4.1}$$

and

$$(F_{v,t_0} - F, z_0) \in \mathcal{I}(\varphi)_{z_0}.$$

Note that

$$\begin{aligned} \left(\int_{D_v} |F_{v,t_0}|^2 d\lambda_n\right)^{\frac{1}{2}} &\leq \left(\int_{D_v} |F_{v,t_0} - (1 - b_{t_0}(\varphi))F|^2 d\lambda_n\right)^{\frac{1}{2}} \\ &\quad + \left(\int_{D_v} |(1 - b_{t_0}(\varphi))F|^2 d\lambda_n\right)^{\frac{1}{2}}, \end{aligned} \quad (4.2)$$

and

$$\lim_{t_0 \rightarrow 0} \int_{D_v} |(1 - b_{t_0}(\varphi))F|^2 d\lambda_n = 0, \quad (4.3)$$

then it follows that

$$\begin{aligned} &\liminf_{t_0 \rightarrow +\infty} \int_{D_v} \mathbb{I}_{\{-t_0 - B_0 < \varphi < -t_0\}} |F|^2 e^{t_0 + B_0} d\lambda_n \\ &\geq \liminf_{t_0 \rightarrow +\infty} \int_{D_v} \mathbb{I}_{\{-t_0 - B_0 < \varphi < -t_0\}} |F|^2 e^{-\varphi} d\lambda_n \\ &\geq \inf\{\|F_1\|_0^2 \mid (F_1 - F, z_0) \in \mathcal{I}(\varphi)_{z_0} \text{ \& } F_1 \in \mathcal{O}_{D_v}\} \end{aligned} \quad (4.4)$$

Proposition 1.10 has thus been proved.

4.2. A proposition used in the proof of Conjecture J-M.

Let

$$\varphi := (1 + \delta) \max\{\psi, \log |F|^2\},$$

and

$$\Psi := \min\{\psi - \log |F|^2, 0\},$$

where δ is a positive integer. Then $\Psi + \varphi$ and $(1 + \delta)\Psi + \varphi$ are both plurisubharmonic functions on Δ^n .

In [20], we proved Theorem 1.11 by the following proposition:

Proposition 4.1. [20] *Let D_v be a strongly pseudoconvex domain relatively compact in Δ^n containing o .*

Then there exists a holomorphic function F_{v,t_0} on D_v , satisfying:

$$(F_{v,t_0} - F^{1+\delta}, o) \in \mathcal{I}(\varphi + \Psi)_o \quad (4.5)$$

and

$$\begin{aligned} &\int_{D_v} |F_{v,t_0} - (1 - b_{t_0}(\Psi))F^{1+\delta}|^2 e^{-\varphi} d\lambda_n \\ &\leq \left(1 + \frac{1}{\delta}\right) \int_{D_v} \left(\frac{1}{B_0} \mathbb{I}_{\{-t_0 - B_0 < t < -t_0\}} \circ \Psi\right) |F^{1+\delta}|^2 e^{-\varphi} e^{-\Psi} d\lambda_n, \end{aligned} \quad (4.6)$$

where

$$b_{t_0} := \int_{-\infty}^t \frac{1}{B_0} \mathbb{I}_{(-t_0 - B_0, -t_0)} ds,$$

and $t_0 \geq 0$.

Replacing the strong pseudoconvexity of D_v by pseudoconvexity, and o by $z_0 \in D_v$ for any v , Lemma 2.1 also holds.

4.3. Proof of Proposition 1.12.

In this subsection, we explicitly point out that the proof of Proposition 1.10 is implicitly contained in our proof of Theorem 1.9 in [20].

For the sake of completeness, we recall our proof of Theorem 1.11 in [20] with slightly modification: changing o into z_0 .

By Proposition 4.1 with $\Psi := \min\{\psi - \log |F|^2, 0\}$, it follows that there exists F_{v,t_0} , which is a holomorphic function on D_v satisfying:

$$\begin{aligned} & \int_{D_v} |F_{v,t_0} - (1 - b_{t_0}(\Psi))F^{1+\delta}|^2 e^{-\varphi} d\lambda_n \\ & \leq (1 + \frac{1}{\delta}) \int_{D_v} \frac{1}{B_0} \mathbb{I}_{\{-t_0 - B_0 < \Psi < -t_0\}} |F^{1+\delta}|^2 e^{-\varphi - \Psi} d\lambda_n, \end{aligned} \quad (4.7)$$

and

$$(F_{v,t_0} - F^{1+\delta}, z_0) \in \mathcal{I}(\varphi + \Psi)_{z_0}.$$

Note that

$$\begin{aligned} (\int_{D_v} |F_{v,t_0}|^2 e^{-\varphi} d\lambda_n)^{\frac{1}{2}} & \leq (\int_{D_v} |F_{v,t_0} - (1 - b_{t_0}(\Psi))F^{1+\delta}|^2 e^{-\varphi} d\lambda_n)^{\frac{1}{2}} \\ & \quad + (\int_{D_v} |(1 - b_{t_0}(\Psi))F^{1+\delta}|^2 e^{-\varphi} d\lambda_n)^{\frac{1}{2}}, \end{aligned} \quad (4.8)$$

and

$$\lim_{t_0 \rightarrow 0} \int_{D_v} |(1 - b_{t_0}(\Psi))F^{1+\delta}|^2 e^{-\varphi} d\lambda_n = 0, \quad (4.9)$$

then it follows that

$$\begin{aligned} & (1 + \frac{1}{\delta}) e^{B_0+1} \frac{1}{B_0} \liminf_{t_0 \rightarrow \infty} \int_{D_v} \mathbb{I}_{\{-t_0 - B_0 < \Psi < -t_0\}} e^{t_0} d\lambda_n \\ & \geq \liminf_{t_0 \rightarrow \infty} (1 + \frac{1}{\delta}) \int_{D_v} \frac{1}{B_0} \mathbb{I}_{\{-t_0 - B_0 < \Psi < -t_0\}} |F^{1+\delta}|^2 e^{-\varphi - \Psi} d\lambda_n \\ & \geq \frac{1}{e^{\sup_D \varphi}} \inf\{\|F_1\|_0^2 \mid (F_1 - F^{1+\delta}, z_0) \in \mathcal{I}(\varphi + \Psi)_{z_0} \text{ \& } F_1 \in \mathcal{O}_{D_v}\}. \end{aligned} \quad (4.10)$$

Note that $\varphi + \Psi = \psi + \delta \max\{\psi, \log |F|^2\}$. Proposition 1.12 has thus been proved.

4.4. Proof of Remark 1.13.

When $D = \Delta$, $F = 1$, $\psi = \log |z|^2$, then I is trivial, $\log |I| = 0$ and $c_o^I(\log |z|^2) = c_o(\log |z|^2) = \frac{1}{2}$.

As $f_i = 1$, then it follows that $\max\{2c_o^I(\psi)\psi, 2\log |f_i|\} = 0$ for any i , and $\max\{2c_o^I(\psi)\psi, 2\log |I|\} = 0$.

It is clear that

$$K_{2c_o^I(\psi)\psi + \delta \max\{2c_o^I(\psi)\psi, \log |f_i|^2\}, F^{1+\delta}}(o) = K_{\psi, 1}(o) = K(o) = \frac{1}{\pi},$$

and

$$\liminf_{r \rightarrow 0} \frac{1}{r^2} \mu(\{c_o^I(\psi)\psi - \log |I| < \log r\}) = \lim_{r \rightarrow 0} \frac{1}{r^2} \mu(\{\log |z| < \log r\}) = \frac{1}{\pi}.$$

When δ goes to ∞ , the equality in inequality 1.12 holds.

4.5. Proof of Remark 1.14.

When $F = 1$, $\psi < 0$, then $f_i = 1$ and $c_o^I(\psi) = c_o(\psi)$.

It is clear that

$$K_{2c_o^I(\psi)\psi+\delta\max\{2c_o^I(\psi)\psi,\log|f_i|^2\},1}^{-1}(o) = K_{2c_o(\psi)\psi,1}^{-1}(o) \geq K^{-1}(o),$$

and

$$\sup_D e^{(1+\delta)\max\{2c_o^I(\psi)\psi,2\log|I|\}} = \sup_D e^{(1+\delta)\max\{2c_o^I(\psi)\psi,0\}} = \sup_D e^0 = 1.$$

Replacing r in inequality 1.12 by $r^{2c_o(\psi)}$, the optimal effectiveness of conjecture J-M degenerates to the optimal effectiveness of conjecture D-K:

$$\begin{aligned} \liminf_{r \rightarrow 0} \frac{1}{r^{2c_o(\psi)}} \mu(\{\psi < \log r\}) &= \liminf_{r \rightarrow 0} \frac{1}{r^2} \mu(\{c_o^I(\psi)\psi - \log|I| < \log r\}) \\ &\geq \sup_{\delta \in \{1,2,\dots\}} \frac{K^{-1}(o)}{1 + \frac{1}{\delta}} = K^{-1}(o), \end{aligned} \quad (4.11)$$

5. PROOFS OF PREPARATORY RESULTS

In this section, we recall some main steps in our proof in [16] (see also [18, 20]) with some slight modifications in order to prove Lemma 2.1.

5.1. Proof of Lemma 2.1.

For the sake of completeness, let's recall some steps in our proof in [16] (see also [18, 20]) with some slight modifications in order to prove Lemma 2.1.

Let $\{v_{t_0,\varepsilon}\}_{t_0 \in \mathbb{R}, \varepsilon \in (0, \frac{1}{8}B_0)}$ be a family of smooth increasing convex functions on \mathbb{R} , which are continuous functions on $\mathbb{R} \cup \{-\infty\}$, such that:

- 1). $v_{t_0,\varepsilon}(t) = t$ for $t \geq -t_0 - \varepsilon$, $v_{t_0,\varepsilon}(t) = \text{constant}$ for $t < -t_0 - B_0 + \varepsilon$;
- 2). $v_{t_0,\varepsilon}''(t)$ are pointwise convergent to $\frac{1}{B_0}\mathbb{I}_{(-t_0-B_0,-t_0)}$, when $\varepsilon \rightarrow 0$, and $0 \leq v_{t_0,\varepsilon}''(t) \leq 2$ for any $t \in \mathbb{R}$;
- 3). $v_{t_0,\varepsilon}'(t)$ are pointwise convergent to $b_{t_0}(t) = \int_{-\infty}^t \frac{1}{B_0}\mathbb{I}_{(-t_0-B_0,-t_0)} ds$ (b_{t_0} is also a continuous function on $\mathbb{R} \cup \{-\infty\}$), when $\varepsilon \rightarrow 0$, and $0 \leq v_{t_0,\varepsilon}'(t) \leq 1$ for any $t \in \mathbb{R}$.

One can construct the family $\{v_{t_0,\varepsilon}\}_{t_0 \in \mathbb{R}, \varepsilon \in (0, \frac{1}{8}B_0)}$ by the setting

$$\begin{aligned} v_{t_0,\varepsilon}(t) &:= \int_{-\infty}^t \left(\int_{-\infty}^{t_1} \left(\frac{1}{1-4\varepsilon} \frac{1}{B_0} \mathbb{I}_{(-t_0-B_0+2\varepsilon,-t_0-2\varepsilon)} * \rho_{\frac{1}{4}\varepsilon} \right)(s) ds \right) dt_1 \\ &\quad - \int_{-\infty}^0 \left(\int_{-\infty}^{t_1} \left(\frac{1}{1-4\varepsilon} \frac{1}{B_0} \mathbb{I}_{(-t_0-B_0+2\varepsilon,-t_0-2\varepsilon)} * \rho_{\frac{1}{4}\varepsilon} \right)(s) ds \right) dt_1, \end{aligned} \quad (5.1)$$

where $\rho_{\frac{1}{4}\varepsilon}$ is the kernel of convolution satisfying $\text{supp}(\rho_{\frac{1}{4}\varepsilon}) \subset (-\frac{1}{4}\varepsilon, \frac{1}{4}\varepsilon)$. Then it follows that

$$v_{t_0,\varepsilon}''(t) = \frac{1}{1-4\varepsilon} \frac{1}{B_0} \mathbb{I}_{(-t_0-B_0+2\varepsilon,-t_0-2\varepsilon)} * \rho_{\frac{1}{4}\varepsilon}(t),$$

and

$$v_{t_0,\varepsilon}'(t) = \int_{-\infty}^t \left(\frac{1}{1-4\varepsilon} \frac{1}{B_0} \mathbb{I}_{(-t_0-B_0+2\varepsilon,-t_0-2\varepsilon)} * \rho_{\frac{1}{4}\varepsilon} \right)(s) ds.$$

As $D_v \subset \subset \Delta^n \subset \mathbb{C}^n$, then there exist negative smooth plurisubharmonic functions $\{\psi_m\}_{m=1,2,\dots}$ on a neighborhood of \overline{D}_v , such that the sequence $\{\psi_m\}_{m=1,2,\dots}$ is decreasingly convergent to ψ on a smaller neighborhood of \overline{D}_v , when $m \rightarrow +\infty$.

Take undetermined functions s and u which will be naturally led to an ODE system after calculations based on the twisted Bochner-Kodaira identity and a lemma of Berndtsson's, and will be determined by solving the ODE system (The idea goes back to [45], [14]).

Let $\eta = s(-v_{t_0,\varepsilon} \circ \psi_m)$ and $\phi = u(-v_{t_0,\varepsilon} \circ \psi_m)$, where $s \in C^\infty((0, +\infty))$ satisfies $s \geq 0$, and $u \in C^\infty((0, +\infty))$ satisfies $\lim_{t \rightarrow +\infty} u(t) = 0$, such that $u''s - s'' > 0$, and $s' - u's = 1$.

Let $\Phi = \psi_m + \phi$.

Now let $\alpha = \sum_{j=1}^n \alpha_j dz^j \in \text{Dom}_{D_v}(\bar{\partial}^*) \cap \text{Ker}(\bar{\partial}) \cap C_{(0,1)}^\infty(\overline{D}_v)$. By Cauchy-Schwarz inequality, it follows that

$$\begin{aligned} 2\text{Re}(\bar{\partial}_\Phi^* \alpha, \alpha_\mathbb{L}(\bar{\partial}\eta)^\sharp)_{\Omega, \Phi} &\geq - \int_{D_v} g^{-1} |\bar{\partial}_\Phi^* \alpha|^2 e^{-\Phi} d\lambda_n \\ &+ \sum_{j,k=1}^n \int_{D_v} (-g(\partial_j \eta) \bar{\partial}_k \eta) \alpha_{\bar{j}} \overline{\alpha_k} e^{-\Phi} d\lambda_n. \end{aligned} \quad (5.2)$$

Using the twisted Bochner-Kodaira identity (see Lemma 3.2. in [45]) and inequality 5.2, since $s \geq 0$ and ψ_m is a plurisubharmonic function on \overline{D}_v , we get

$$\begin{aligned} &\int_{D_v} (\eta + g^{-1}) |\bar{\partial}_\Phi^* \alpha|^2 e^{-\Phi} d\lambda_n \\ &\geq \sum_{j,k=1}^n \int_{D_v} (-\partial_j \bar{\partial}_k \eta + \eta \partial_j \bar{\partial}_k \phi - g(\partial_j \eta) \bar{\partial}_k \eta) \alpha_{\bar{j}} \overline{\alpha_k} e^{-\Phi} d\lambda_n, \end{aligned} \quad (5.3)$$

where g is a positive continuous function on D_v . We need some calculations to determine g .

We have

$$\begin{aligned} &\sum_{1 \leq j, k \leq n} (-\partial_j \bar{\partial}_k \eta + \eta \partial_j \bar{\partial}_k \phi - g(\partial_j \eta) \bar{\partial}_k \eta) \alpha_{\bar{j}} \overline{\alpha_k} \\ &= (s' - su') \sum_{1 \leq j, k \leq n} ((v'_{t_0,\varepsilon} \circ \psi_m) \partial_j \bar{\partial}_k \psi_m + (v''_{t_0,\varepsilon} \circ \psi_m) \partial_j (\psi_m) \bar{\partial}_k (\psi_m)) \alpha_{\bar{j}} \overline{\alpha_k} \\ &+ ((u''s - s'') - gs'^2) \sum_{1 \leq j, k \leq n} \partial_j (-v_{t_0,\varepsilon} \circ \psi_m) \bar{\partial}_k (-v_{t_0,\varepsilon} \circ \psi_m) \alpha_{\bar{j}} \overline{\alpha_k}. \end{aligned} \quad (5.4)$$

We omit composite item $(-v_{t_0,\varepsilon} \circ \psi_m)$ after $s' - su'$ and $(u''s - s'') - gs'^2$ in the above equalities.

Set

$$g = \frac{u''s - s''}{s'^2} \circ (-v_{t_0,\varepsilon} \circ \psi_m).$$

It follows that

$$\eta + g^{-1} = \left(s + \frac{s'^2}{u''s - s''}\right) \circ (-v_{t_0,\varepsilon} \circ \psi_m).$$

Because of $v'_{t_0, \varepsilon} \geq 0$ and $s' - su' = 1$, using inequalities 5.3, we have

$$\int_{D_v} (\eta + g^{-1}) |\bar{\partial}_{\Phi}^* \alpha|^2 e^{-\Phi} d\lambda_n \geq \int_{D_v} (v''_{t_0, \varepsilon} \circ \psi_m) |\alpha_{\perp} (\bar{\partial} \psi_m)^{\sharp}|^2 e^{-\Phi} d\lambda_n. \quad (5.5)$$

Let $\lambda = \bar{\partial}[(1 - v'_{t_0, \varepsilon}(\psi_m))F]$. By the definition of contraction, Cauchy-Schwarz inequality and inequality 5.5, it follows that

$$\begin{aligned} |(\lambda, \alpha)_{D_v, \Phi}|^2 &= |((v''_{t_0, \varepsilon} \circ \psi_m) \bar{\partial} \psi_m F, \alpha)_{D_v, \Phi}|^2 \\ &= |((v''_{t_0, \varepsilon} \circ \psi_m) F, \alpha_{\perp} (\bar{\partial} \psi_m)^{\sharp})_{D_v, \Phi}|^2 \\ &\leq \int_{D_v} (v''_{t_0, \varepsilon} \circ \psi_m) |F|^2 e^{-\Phi} d\lambda_n \int_{D_v} (v''_{t_0, \varepsilon} \circ \psi_m) |\alpha_{\perp} (\bar{\partial} \psi_m)^{\sharp}|^2 e^{-\Phi} d\lambda_n \\ &\leq \left(\int_{D_v} (v''_{t_0, \varepsilon} \circ \psi_m) |F|^2 e^{-\Phi} d\lambda_n \right) \left(\int_{D_v} (\eta + g^{-1}) |\bar{\partial}_{\Phi}^* \alpha|^2 e^{-\Phi} d\lambda_n \right). \end{aligned} \quad (5.6)$$

Let $\mu := (\eta + g^{-1})^{-1}$. Using a lemma of Berndtsson's (see [1] or Lemma 3.7. in [45]), we have locally L^1 function $u_{v, t_0, m, \varepsilon}$ on D_v such that $\bar{\partial} u_{v, t_0, m, \varepsilon} = \lambda$, and

$$\int_{D_v} |u_{v, t_0, m, \varepsilon}|^2 (\eta + g^{-1})^{-1} e^{-\Phi} d\lambda_n \leq \int_{D_v} (v''_{t_0, \varepsilon} \circ \psi_m) |F|^2 e^{-\Phi} d\lambda_n. \quad (5.7)$$

Let $\mu_1 = e^{v_{t_0, \varepsilon} \circ \psi_m}$, $\tilde{\mu} = \mu_1 e^{\phi}$. Assume that we can choose η and ϕ such that $\tilde{\mu} \leq \mathbf{C}(\eta + g^{-1})^{-1} = \mu$, where $\mathbf{C} = 1$.

Note that $v_{t_0, \varepsilon}(\psi_m) \geq \psi_m$. Then it follows that

$$\int_{D_v} |u_{v, t_0, m, \varepsilon}|^2 d\lambda_n \leq \int_{D_v} |u_{v, t_0, m, \varepsilon}|^2 \mu_1 e^{\phi} e^{-\psi_m - \phi} d\lambda_n = \int_{D_v} |u_{v, t_0, m, \varepsilon}|^2 \tilde{\mu} e^{-\Phi} d\lambda_n. \quad (5.8)$$

Using inequalities 5.7 and 5.8, we obtain that

$$\int_{D_v} |u_{v, t_0, m, \varepsilon}|^2 d\lambda_n \leq \mathbf{C} \int_{D_v} (v''_{t_0, \varepsilon} \circ \psi_m) |F|^2 e^{-\Phi} d\lambda_n,$$

under the assumption $\tilde{\mu} \leq \mathbf{C}(\eta + g^{-1})^{-1}$.

As $-v_{t_0, \varepsilon} \circ \psi_m(\overline{D_v}) \subset \subset (0, t_0 + 1)$ and $\{\psi_m\}_{m=1, 2, \dots}$ is decreasing, then it is clear that

$$-v_{t_0, \varepsilon} \circ \psi_m(\overline{D_v}) \subset \subset K_{t_0} \subset \subset (0, t_0 + 1) \quad (5.9)$$

where K_{t_0} is independent of m and $\varepsilon \in (0, \frac{1}{8}B_0)$.

As u is positive and smooth on $(0, +\infty)$, it follows that ϕ is uniformly bounded on $\overline{D_v}$ independent of m .

As $\text{Supp}(v''_{t_0, \varepsilon}) \subset \subset (-t_0 - B_0, -t_0)$, then it is clear that $(v''_{t_0, \varepsilon} \circ \psi_m) |F|^2 e^{-\psi_m}$ are uniformly bounded on $\overline{D_v}$ independent of m .

Therefore

$$\int_{D_v} (v''_{t_0, \varepsilon} \circ \psi_m) |F|^2 e^{-\Phi} d\lambda_n$$

are uniformly bounded independent of m , for any given v, t_0, ε .

By weakly compactness of the unit ball of $L^2(D_v)$ and dominated convergence theorem, when $m \rightarrow +\infty$, it follows that the weak limit of some weakly convergent subsequence of $\{u_{v, t_0, m, \varepsilon}\}_m$ gives $u_{v, t_0, \varepsilon}$ on D_v satisfying

$$\int_{D_v} |u_{v, t_0, \varepsilon}|^2 d\lambda_n \leq \frac{\mathbf{C}}{e^{A_{t_0}}} \int_{D_v} (v''_{t_0, \varepsilon} \circ \psi) |F|^2 e^{-\psi} d\lambda_n, \quad (5.10)$$

where $A_{t_0} := \inf_{t \in (t_0, t_0 + B_0)} \{u(t)\}$.

As ψ_m is decreasingly convergent to ψ on Δ^n , and $\psi(o) = -\infty$, then for any given t_0 there exists m_0 and a neighbourhood U_0 of $o \in D_v$ on Δ^n , such that for any $m \geq m_0$ and $\varepsilon \in (0, \frac{1}{8}B_0)$, $v''_{t_0, \varepsilon} \circ \psi_m|_{U_0} = 0$.

It follows that

$$\bar{\partial}u_{v, t_0, m, \varepsilon}|_{U_0} = \lambda|_{U_0} = \bar{\partial}[(1 - v'_{t_0, \varepsilon}(\psi_m))F]|_{U_0} = -(v''_{t_0, \varepsilon} \circ \psi_m)F\bar{\partial}\psi_m|_{U_0} = 0.$$

That is to say $u_{v, t_0, m, \varepsilon}|_{U_0}$ are all holomorphic. Therefore $u_{v, t_0, \varepsilon}|_{U_0}$ is holomorphic.

Recall that the integrals $\int_{D_v} |u_{v, t_0, m, \varepsilon}|^2 d\lambda_n$ have a uniform bound independent of m , then we can choose a subsequence with respect to m from the chosen weakly convergent subsequence of $u_{v, t_0, m, \varepsilon}$, such that the subsequence is uniformly convergent on any compact subset of U_0 , and we still denote the subsequence by $u_{v, t_0, m, \varepsilon}$ without ambiguity.

By the above arguments, it follows that the right hand side of inequality 5.7 are uniformly bounded independent of m and $\varepsilon \in (0, \frac{1}{8}B_0)$.

By inequality 5.7, it follows that

$$\int_{D_v} |u_{v, t_0, m, \varepsilon}|^2 (\eta + g^{-1})^{-1} e^{-\phi - \psi_m} d\lambda_n$$

are uniformly bounded independent of m and $\varepsilon \in (0, \frac{1}{8}B_0)$.

Using inequality 5.9, we obtain that

$$(\eta + g^{-1})^{-1} = (s(-v_{t_0, \varepsilon} \circ \psi_m) + \frac{s'^2}{u''s - s''} \circ (-v_{t_0, \varepsilon} \circ \psi_m))^{-1}$$

and $e^{-\phi} = e^{-u(-v_{t_0, \varepsilon} \circ \psi_m)}$ have positive uniform lower bounds independent of m and $\varepsilon \in (0, \frac{1}{8}B_0)$.

Then the integrals

$$\int_{K_0} |u_{v, t_0, m, \varepsilon}|^2 e^{-\psi_m} d\lambda_n$$

have a uniform upper bound independent of m and $\varepsilon \in (0, \frac{1}{8}B_0)$ for any given compact set $K_0 \subset \subset U_0 \cap D_v$, and

$$\bar{\partial}u_{v, t_0, m, \varepsilon}|_{U_0} = 0.$$

As $\psi_{m'} \leq \psi_m$ where $m' \geq m$, it follows that

$$|u_{v, t_0, m', \varepsilon}|^2 e^{-\psi_m} \leq |u_{v, t_0, m', \varepsilon}|^2 e^{-\psi_{m'}}.$$

Then for any given compact set $K_0 \subset \subset U_0 \cap D_v$, $\int_{K_0} |u_{v, t_0, m', \varepsilon}|^2 e^{-\psi_m} d\lambda_n$ have a uniform bound independent of m and m' .

It is clear that for any given compact set $K_0 \subset \subset U_0 \cap D_v$, the integrals

$$\int_{K_0} |u_{v, t_0, \varepsilon}|^2 e^{-\psi_m} d\lambda_n$$

have a uniform bound independent of m and $\varepsilon \in (0, \frac{1}{8}B_0)$.

Therefore the integrals $\int_{K_0} |u_{v, t_0, \varepsilon}|^2 e^{-\psi} d\lambda_n$ have a uniform bound independent of $\varepsilon \in (0, \frac{1}{8}B_0)$, for any given compact set $K_0 \subset \subset U_0 \cap D_v$ containing o .

In summary, we have $|u_{v, t_0, \varepsilon}|^2 e^{-\psi}$ is integrable near o , and $\bar{\partial}u_{v, t_0, \varepsilon} = 0$ near o . That is to say

$$(u_{v, t_0, \varepsilon}, o) \in \mathcal{I}(\psi)_o.$$

Let $F_{v,t_0,\varepsilon} := (1 - v'_{t_0,\varepsilon} \circ \psi)F - u_{v,t_0,\varepsilon}$. By inequality 5.10 and $(u_{v,t_0,\varepsilon}, o) \in \mathcal{I}(\psi)_o$, it follows that $F_{v,t_0,\varepsilon}$ is a holomorphic function on D_v satisfying $(F_{v,t_0,\varepsilon} - F, o) \in \mathcal{I}(\psi)_o$ and

$$\begin{aligned} & \int_{D_v} |F_{v,t_0,\varepsilon} - (1 - v'_{t_0,\varepsilon} \circ \psi)F|^2 d\lambda_n \\ & \leq \frac{\mathbf{C}}{e^{At_0}} \int_{D_v} (v''_{t_0,\varepsilon} \circ \psi) |F|^2 e^{-\psi} d\lambda_n. \end{aligned} \quad (5.11)$$

Given t_0 and D_v , it is clear that $(v''_{t_0,\varepsilon} \circ \psi) |F|^2 e^{-\psi}$ have a uniform bound on D_v independent of ε .

Then the integrals $\int_{D_v} (v''_{t_0,\varepsilon} \circ \psi) |F|^2 e^{-\psi} d\lambda_n$ have a uniform bound independent of ε , for any given t_0 and D_v .

As $|(1 - v'_{t_0,\varepsilon} \circ \psi)F|^2$ have a uniform bound on D_v independent of ε , it follows that the integrals $\int_{D_v} |(1 - v'_{t_0,\varepsilon} \circ \psi)F|^2 d\lambda_n$ have a uniform bound independent of ε , for any given t_0 and D_v .

As

$$\begin{aligned} & \int_{D_v} |F_{v,t_0,\varepsilon}|^2 d\lambda_n \\ & \leq \int_{D_v} |F_{v,t_0,\varepsilon} - (1 - v'_{t_0,\varepsilon} \circ \psi)F|^2 d\lambda_n + \int_{D_v} |(1 - v'_{t_0,\varepsilon} \circ \psi)F|^2 d\lambda_n \\ & \leq \frac{\mathbf{C}}{e^{At_0}} \int_{D_v} (v''_{t_0,\varepsilon} \circ \psi) |F|^2 e^{-\psi} d\lambda_n + \int_{D_v} |(1 - v'_{t_0,\varepsilon} \circ \psi)F|^2 d\lambda_n, \end{aligned} \quad (5.12)$$

then $\int_{D_v} |F_{v,t_0,\varepsilon}|^2 d\lambda_n$ have a uniform bound independent of ε .

As $\partial F_{v,t_0,\varepsilon} = 0$ when $\varepsilon \rightarrow 0$ and the unit ball of $L^2(D_v)$ is weakly compact, it follows that the weak limit of some weakly convergent subsequence of $\{F_{v,t_0,\varepsilon}\}_\varepsilon$ gives us a holomorphic function F_{v,t_0} on Δ^n .

Then we can also choose a subsequence of the weakly convergent subsequence of $\{F_{v,t_0,\varepsilon}\}_\varepsilon$, such that the chosen sequence is uniformly convergent on any compact subset of D_v , denoted by $\{F_{v,t_0,\varepsilon}\}_\varepsilon$ without ambiguity.

For any given compact subset K_0 on D_v , $F_{v,t_0,\varepsilon}$, $(1 - v'_{t_0,\varepsilon} \circ \psi)F$ and $(v''_{t_0,\varepsilon} \circ \psi) |F|^2 e^{-\psi}$ have uniform bounds on K_0 independent of ε .

As the integrals $\int_{K_0} |u_{v,t_0,\varepsilon}|^2 e^{-\psi} d\lambda_n$ have a uniform bound independent of $\varepsilon \in (0, \frac{1}{8}B_0)$, for any given compact set $K_0 \subset \subset U_0 \cap D_v$ containing o , it follows that

$$(F_{v,t_0} - (1 - b_{t_0}(\psi))F, o) \in \mathcal{I}(\psi)_o.$$

Using the dominated convergence theorem on any compact subset K of D_v and inequality 5.11, we obtain

$$\begin{aligned} & \int_K |F_{v,t_0} - (1 - b_{t_0}(\psi))F|^2 d\lambda_n \\ & \leq \frac{\mathbf{C}}{e^{At_0}} \int_{D_v} \left(\frac{1}{B_0} \mathbb{I}_{\{-t_0 - B_0 < t < -t_0\}} \circ \psi \right) |F|^2 e^{-\psi} d\lambda_n. \end{aligned} \quad (5.13)$$

It suffices to find η and ϕ such that $(\eta + g^{-1}) \leq \mathbf{C}e^{-\psi_m}e^{-\phi} = \mathbf{C}\check{\mu}^{-1}$ on D_v . As $\eta = s(-v_{t_0,\varepsilon} \circ \psi_m)$ and $\phi = u(-v_{t_0,\varepsilon} \circ \psi_m)$, we have $(\eta + g^{-1})e^{v_{t_0,\varepsilon} \circ \psi_m}e^\phi = (s + \frac{s'^2}{u's - s''})e^{-t}e^u \circ (-v_{t_0,\varepsilon} \circ \psi_m)$.

Summarizing the above discussion about s and u , we are naturally led to a system of ODEs (see [14, 16, 18, 20]):

$$\begin{aligned} 1). & \left(s + \frac{s'^2}{u''s - s''}\right)e^{u-t} = \mathbf{C}, \\ 2). & s' - su' = 1, \end{aligned} \quad (5.14)$$

where $t \in [0, +\infty)$, and $\mathbf{C} = 1$.

It is not hard to solve the ODE system 5.14 and get $u = -\log(1 - e^{-t})$ and $s = \frac{t}{1-e^{-t}} - 1$. It follows that $s \in C^\infty((0, +\infty))$ satisfies $s \geq 0$, $\lim_{t \rightarrow +\infty} u(t) = 0$ and $u \in C^\infty((0, +\infty))$ satisfies $u''s - s'' > 0$.

As $u = -\log(1 - e^{-t})$ is decreasing with respect to t , then

$$\begin{aligned} \frac{\mathbf{C}}{e^{At_0}} &= \frac{1}{e^{\inf_{t \in (t_0+B_0, t_0)} u(t)}} \\ &= \sup_{t \in (t_0+B_0, t_0)} \frac{1}{e^{u(t)}} = \sup_{t \in (t_0+B_0, t_0)} (1 - e^{-t}) = 1 - e^{-(t_0+B_0)}, \end{aligned} \quad (5.15)$$

therefore we are done. Thus we prove Lemma 2.1.

5.2. Proof of Proposition 4.1.

Let

$$\varphi := (1 + \delta) \max\{\psi, \log |F|^2\}.$$

It suffices to prove the case that

$$\Psi := \min\{\psi - \log |F|^2, 0\} - a,$$

where $a > 0$ is arbitrarily given.

For the sake of completeness, we recall our proof in [16] (see also [18]) with slightly modifications.

As $D_v \subset \subset \Delta^n \subset \mathbb{C}^n$, then there exist negative smooth plurisubharmonic functions $\{\varphi_m\}_{m=1,2,\dots}$ and smooth functions $\{\Psi_m\}_{m=1,2,\dots}$ on a neighborhood of \overline{D}_v , such that

- 1). $\{\varphi_m + \Psi_m\}_{m=1,2,\dots}$ and $\{\varphi_m + (1 + \delta)\Psi_m\}_{m=1,2,\dots}$ are negative smooth plurisubharmonic functions;
 - 2). the sequence $\{\varphi_m\}_{m=1,2,\dots}$ is decreasingly convergent to φ ;
 - 3). the sequence $\{\varphi_m + \Psi_m\}_{m=1,2,\dots}$ is decreasingly convergent to $\varphi + \Psi$;
- on a smaller neighborhood of \overline{D}_v , when $m \rightarrow +\infty$.

Let $\eta = s(-v_{t_0, \varepsilon} \circ \Psi_m)$ and $\phi = u(-v_{t_0, \varepsilon} \circ \Psi_m)$, where $s \in C^\infty((0, +\infty))$ satisfies $s \geq \frac{1}{\delta}$, and $u \in C^\infty((0, +\infty))$, such that $u''s - s'' > 0$, and $s' - u's = 1$.

Let $\Phi := \varphi_m + \Psi_m + \phi$.

Now let $\alpha = \sum_{j=1}^n \alpha_j d\bar{z}^j \in \text{Dom}_{D_v}(\bar{\partial}^*) \cap \text{Ker}(\bar{\partial}) \cap C_{(0,1)}^\infty(\overline{D}_v)$. By Cauchy-Schwarz inequality, it follows that

$$\begin{aligned} 2\text{Re}(\bar{\partial}_\Phi^* \alpha, \alpha_\mathbb{L}(\bar{\partial}\eta)^\sharp)_{\Omega, \Phi} &\geq - \int_{D_v} g^{-1} |\bar{\partial}_\Phi^* \alpha|^2 e^{-\Phi} d\lambda_n \\ &\quad + \sum_{j,k=1}^n \int_{D_v} (-g(\partial_j \eta) \bar{\partial}_k \eta) \alpha_{\bar{j}} \overline{\alpha_k} e^{-\Phi} d\lambda_n. \end{aligned} \quad (5.16)$$

Using the twisted Bochner-Kodaira identity (see Lemma 3.2. in [45]) and inequality 5.16, since $s \geq 0$ and φ_m is a plurisubharmonic function on \overline{D}_v , we get

$$\begin{aligned}
& \int_{D_v} (\eta + g^{-1}) |\bar{\partial}_\Phi^* \alpha|^2 e^{-\Phi} d\lambda_n \\
& \geq \sum_{j,k=1}^n \int_{D_v} (-\partial_j \bar{\partial}_k \eta + \eta \partial_j \bar{\partial}_k \phi + \eta \partial_j \bar{\partial}_k (\Psi_m + \varphi_m) - g(\partial_j \eta) \bar{\partial}_k \eta) \alpha_{\bar{j}} \bar{\alpha}_{\bar{k}} e^{-\Phi} d\lambda_n,
\end{aligned} \tag{5.17}$$

where g is a positive continuous function on D_v .

We need some calculations to determine g .

We have

$$\begin{aligned}
& \sum_{1 \leq j, k \leq n} (-\partial_j \bar{\partial}_k \eta + \eta \partial_j \bar{\partial}_k \phi - g(\partial_j \eta) \bar{\partial}_k \eta) \alpha_{\bar{j}} \bar{\alpha}_{\bar{k}} \\
& = (s' - su') \sum_{1 \leq j, k \leq n} \partial_j \bar{\partial}_k (v_{t_0, \varepsilon} \circ \Psi_m) \alpha_{\bar{j}} \bar{\alpha}_{\bar{k}} \\
& \quad + ((u''s - s'') - gs'^2) \sum_{1 \leq j, k \leq n} \partial_j (-v_{t_0, \varepsilon} \circ \Psi_m) \bar{\partial}_k (-v_{t_0, \varepsilon} \circ \Psi_m) \alpha_{\bar{j}} \bar{\alpha}_{\bar{k}} \\
& = (s' - su') \sum_{1 \leq j, k \leq n} ((v'_{t_0, \varepsilon} \circ \Psi_m) \partial_j \bar{\partial}_k \Psi_m + (v''_{t_0, \varepsilon} \circ \Psi_m) \partial_j (\Psi_m) \bar{\partial}_k (\Psi_m)) \alpha_{\bar{j}} \bar{\alpha}_{\bar{k}} \\
& \quad + ((u''s - s'') - gs'^2) \sum_{1 \leq j, k \leq n} \partial_j (-v_{t_0, \varepsilon} \circ \Psi_m) \bar{\partial}_k (-v_{t_0, \varepsilon} \circ \Psi_m) \alpha_{\bar{j}} \bar{\alpha}_{\bar{k}}.
\end{aligned} \tag{5.18}$$

We omit composite item $(-v_{t_0, \varepsilon} \circ \Psi_m)$ after $s' - su'$ and $(u''s - s'') - gs'^2$ in the above equalities.

Since $\varphi_m + \Psi_m$ and $\varphi_m + (1 + \delta)\Psi_m$ are plurisubharmonic on \bar{D}_v and $0 \leq v'_{t_0, \varepsilon} \circ \Psi_m \leq 1$, we have

$$(1 - v'_{t_0, \varepsilon} \circ \Psi_m) \sqrt{-1} \partial \bar{\partial} (\varphi_m + \Psi_m) + (v'_{t_0, \varepsilon} \circ \Psi_m) \sqrt{-1} \partial \bar{\partial} (\varphi_m + (1 + \delta)\Psi_m) \geq 0, \tag{5.19}$$

on \bar{D}_v , which means that

$$\frac{1}{\delta} \sqrt{-1} \partial \bar{\partial} (\varphi_m + \Psi_m) + (v'_{t_0, \varepsilon} \circ \Psi_m) \sqrt{-1} \partial \bar{\partial} \Psi_m \geq 0, \tag{5.20}$$

on \bar{D}_v .

Let $g = \frac{u''s - s''}{s'^2} \circ (-v_{t_0, \varepsilon} \circ \Psi_m)$. It follows that $\eta + g^{-1} = (s + \frac{s'^2}{u''s - s''}) \circ (-v_{t_0, \varepsilon} \circ \Psi_m)$.

Because of $v'_{t_0, \varepsilon} \geq 0$ and $s' - su' = 1$, using inequalities 5.17 5.20 and 5.18, we have

$$\int_{D_v} (\eta + g^{-1}) |\bar{\partial}_\Phi^* \alpha|^2 e^{-\Phi} d\lambda_n \geq \int_{D_v} (v''_{t_0, \varepsilon} \circ \Psi_m) |\alpha_{\mathbb{L}} (\bar{\partial} \Psi_m)^\sharp|^2 e^{-\Phi} d\lambda_n. \tag{5.21}$$

Let $\lambda = \bar{\partial}[(1 - v'_{t_0, \varepsilon}(\Psi_m))F^{1+\delta}]$. By the definition of contraction, Cauchy-Schwarz inequality and inequality 5.21, it follows that

$$\begin{aligned}
& |(\lambda, \alpha)_{D_v, \Phi}|^2 = |((v''_{t_0, \varepsilon} \circ \Psi_m)F^{1+\delta}, \alpha_{\mathbb{L}}(\bar{\partial} \Psi_m)^\sharp)_{D_v, \Phi}|^2 \\
& \leq \left(\int_{D_v} (v''_{t_0, \varepsilon} \circ \Psi_m) |F^{1+\delta}|^2 e^{-\Phi} d\lambda_n \right) \left(\int_{D_v} (\eta + g^{-1}) |\bar{\partial}_\Phi^* \alpha|^2 e^{-\Phi} d\lambda_n \right).
\end{aligned} \tag{5.22}$$

Let $\mu := (\eta + g^{-1})^{-1}$. Using a lemma of Berndtsson's (see [1] or Lemma 3.7 in [45]), we have locally L^1 function $u_{v,t_0,m,\varepsilon}$ on D_v such that $\bar{\partial}u_{v,t_0,m,\varepsilon} = \lambda$, and

$$\int_{D_v} |u_{v,t_0,m,\varepsilon}|^2 (\eta + g^{-1})^{-1} e^{-\Phi} d\lambda_n \leq \int_{D_v} (v''_{t_0,\varepsilon} \circ \Psi_m) |F^{1+\delta}|^2 e^{-\Phi} d\lambda_n. \quad (5.23)$$

Let $\mu_1 = e^{v_{t_0,\varepsilon} \circ \Psi_m}$, $\tilde{\mu} = \mu_1 e^\phi$. Assume that we can choose η and ϕ such that $\tilde{\mu} \leq \mathbf{C}(\eta + g^{-1})^{-1} = \mu$, where $\mathbf{C} = 1$.

Note that $v_{t_0,\varepsilon}(\Psi_m) \geq \Psi_m$. Then it follows that

$$\int_{D_v} |u_{v,t_0,m,\varepsilon}|^2 e^{-\varphi_m} d\lambda_n \leq \int_{D_v} |u_{v,t_0,m,\varepsilon}|^2 \tilde{\mu} e^{-\Phi} d\lambda_n. \quad (5.24)$$

Using inequalities 5.23 and 5.24, we obtain that

$$\int_{D_v} |u_{v,t_0,m,\varepsilon}|^2 e^{-\varphi_m} d\lambda_n \leq \mathbf{C} \int_{D_v} (v''_{t_0,\varepsilon} \circ \Psi_m) |F^{1+\delta}|^2 e^{-\Phi} d\lambda_n,$$

under the assumption $\tilde{\mu} \leq \mathbf{C}(\eta + g^{-1})^{-1}$.

As $-v_{t_0,\varepsilon} \circ \Psi_m(\overline{D_v}) \subset \subset (-\infty, t_0 + 1)$, then it is clear that

$$-v_{t_0,\varepsilon} \circ \Psi_m(\overline{D_v}) \subset (-\infty, K_{t_0}) \quad (5.25)$$

where K_{t_0} is independent of m and $\varepsilon \in (0, \frac{1}{8}B_0)$.

As u is positive and smooth on $(-\infty, +\infty)$, it follows that ϕ is uniformly bounded on $\overline{D_v}$ independent of m .

As $\text{Supp}(v''_{t_0,\varepsilon}) \subset \subset (-t_0 - B_0, -t_0)$ and

$$|F^{1+\delta}|^2 e^{-\varphi_m} \leq |F^{1+\delta}|^2 e^{-\varphi} \leq e^{-(1+\delta) \max\{\psi - \log |F|^2, 0\}} \leq 1,$$

then it is clear that $(v''_{t_0,\varepsilon} \circ \Psi_m) |F^{1+\delta}|^2 e^{-\varphi_m - \Psi_m}$ are uniformly bounded on $\overline{D_v}$ independent of m .

Therefore the integrals $\int_{D_v} (v''_{t_0,\varepsilon} \circ \Psi_m) |F^{1+\delta}|^2 e^{-\Phi} d\lambda_n$ are uniformly bounded independent of m , for any given v, t_0, ε .

By weakly compactness of the unit ball of $L^2_\varphi(D_v)$ and dominated convergence theorem, when $m \rightarrow +\infty$, it follows that the weak limit of some weakly convergent subsequence of $\{u_{v,t_0,m,\varepsilon}\}_m$ gives function $u_{v,t_0,\varepsilon}$ on D_v satisfying

$$\int_{D_v} |u_{v,t_0,\varepsilon}|^2 e^{-\varphi} d\lambda_n \leq \frac{\mathbf{C}}{e^{A_{t_0}}} \int_{D_v} (v''_{t_0,\varepsilon} \circ \Psi) |F^{1+\delta}|^2 e^{-\varphi - \Psi} d\lambda_n, \quad (5.26)$$

where $A_{t_0} := \inf_{t \geq t_0} \{u(t)\}$.

Let $F_{v,t_0,m,\varepsilon} := (1 - v'_{t_0,\varepsilon}(\Psi_m)) F^{1+\delta} - u_{v,t_0,m,\varepsilon}$, which is a holomorphic function on D_v .

As $|F^{1+\delta}|^2 e^{-\varphi_m} \leq |F^{1+\delta}|^2 e^{-\varphi} \leq 1$, then the integrals

$$\int_{D_v} |(1 - v'_{t_0,\varepsilon}(\Psi_m)) F^{1+\delta}|^2 e^{-\varphi_m} d\lambda_n$$

have a uniform bound independent of m . Recall that the integrals

$$\int_{D_v} |u_{v,t_0,m,\varepsilon}|^2 e^{-\varphi_m} d\lambda_n$$

have a uniform bound independent of m , then the integrals

$$\int_{D_v} |F_{v,t_0,m,\varepsilon}|^2 e^{-\varphi_m} d\lambda_n$$

have a uniform bound independent of m .

Therefore we can choose a subsequence of $\{F_{v,t_0,m,\varepsilon}\}_{m=1,2,\dots}$ from the chosen weakly convergent subsequence of

$$(1 - v'_{t_0,\varepsilon}(\Psi_m))F^{1+\delta} - u_{v,t_0,m,\varepsilon},$$

such that the subsequence is uniformly convergent on any compact subset of D_v , and we still denote the subsequence by $\{F_{v,t_0,m,\varepsilon}\}_{m=1,2,\dots}$ without ambiguity.

Denote by

$$F_{v,t_0,\varepsilon} := \lim_{m \rightarrow \infty} F_{v,t_0,m,\varepsilon}.$$

By the above arguments, it follows that the right hand side of inequality 5.23 are uniformly bounded independent of m and $\varepsilon \in (0, \frac{1}{8}B_0)$.

By inequality 5.23, it follows that the integrals

$$\int_{D_v} |u_{v,t_0,m,\varepsilon}|^2 (\eta + g^{-1})^{-1} e^{-\phi - \varphi_m - \Psi_m} d\lambda_n$$

are uniformly bounded independent of m and $\varepsilon \in (0, \frac{1}{8}B_0)$.

Using inequality 5.25, we obtain that

$$(\eta + g^{-1})^{-1} = (s(-v_{t_0,\varepsilon} \circ \Psi_m) + \frac{s'^2}{u''s - s''} \circ (-v_{t_0,\varepsilon} \circ \Psi_m))^{-1}$$

and $e^{-\phi} = e^{-u(-v_{t_0,\varepsilon} \circ \Psi_m)}$ have positive uniform lower bounds independent of m and $\varepsilon \in (0, \frac{1}{8}B_0)$.

Then the integrals

$$\int_{K_0} |u_{v,t_0,m,\varepsilon}|^2 e^{-\varphi_m - \Psi_m} d\lambda_n$$

have a uniform upper bound independent of m and $\varepsilon \in (0, \frac{1}{8}B_0)$ for any given compact set $K_0 \subset \subset D_v$.

As

$$\text{Supp}(v'_{t_0,\varepsilon}(\Psi_m)) \subset \{\Psi_m > -t_0 - 1\},$$

it follows that

$$|v'_{t_0,\varepsilon}(\Psi_m)|^2 e^{-\Psi_m} \leq e^{t_0+1}.$$

Furthermore, as

$$|F^{1+\delta}|^2 e^{-\varphi_m} \leq |F^{1+\delta}|^2 e^{-\varphi} = e^{-(1+\delta) \max\{\psi - \log |F|^2, 0\}} \leq 1$$

then the integrals

$$\int_{K_0} |v'_{t_0,\varepsilon}(\Psi_m) F^{1+\delta}|^2 e^{-\varphi_m - \Psi_m} d\lambda_n$$

have a uniform upper bound independent of m and $\varepsilon \in (0, \frac{1}{8}B_0)$ for any given compact set $K_0 \subset \subset D_v$. Therefore the integrals

$$\int_{K_0} |F_{v,t_0,m,\varepsilon} - F^{1+\delta}|^2 e^{-\varphi_m - \Psi_m} d\lambda_n \quad (5.27)$$

have a uniform upper bound independent of m and $\varepsilon \in (0, \frac{1}{8}B_0)$ for any given compact set $K_0 \subset \subset D_v$.

As $\varphi_{m'} + \Psi_{m'} \leq \varphi_m + \Psi_m$ where $m' \geq m$, it follows that

$$|F_{v,t_0,m',\varepsilon} - F^{1+\delta}|^2 e^{-(\varphi_m + \Psi_m)} \leq |F_{v,t_0,m',\varepsilon} - F^{1+\delta}|^2 e^{-(\varphi_{m'} + \Psi_{m'})}.$$

By inequality 5.27, it follows that for any given compact set $K_0 \subset \subset D_v$, the integrals

$$\int_{K_0} |F_{v,t_0,m',\varepsilon} - F^{1+\delta}|^2 e^{-(\varphi_m + \Psi_m)} d\lambda_n$$

have a uniform bound independent of m and m' .

Therefore for any given compact set $K_0 \subset \subset \cap D_v$, the integrals

$$\int_{K_0} |F_{v,t_0,\varepsilon} - F^{1+\delta}|^2 e^{-(\varphi_m + \Psi_m)} d\lambda_n$$

have a uniform bound independent of m and $\varepsilon \in (0, \frac{1}{8}B_0)$.

It is clear that the integrals

$$\int_{K_0} |F_{v,t_0,\varepsilon} - F^{1+\delta}|^2 e^{-\varphi - \Psi} d\lambda_n \quad (5.28)$$

have a uniform upper bound independent of $\varepsilon \in (0, \frac{1}{8}B_0)$ for any given compact set $K_0 \subset \subset D_v$.

In summary, we have $|F_{v,t_0,\varepsilon} - F^{1+\delta}|^2 e^{-\varphi - \Psi}$ is integrable near o . That is to say

$$(F_{v,t_0,\varepsilon} - F^{1+\delta}, o) \in \mathcal{I}(\varphi + \Psi)_o.$$

By inequality 5.26, it follows that $F_{v,t_0,\varepsilon}$ is a holomorphic function on D_v satisfying $(F_{v,t_0,\varepsilon} - F^{1+\delta}, o) \in \mathcal{I}(\varphi)_o$ and

$$\begin{aligned} & \int_{D_v} |F_{v,t_0,\varepsilon} - (1 - v'_{t_0,\varepsilon} \circ \Psi)F^{1+\delta}|^2 e^{-\varphi} d\lambda_n \\ & \leq \frac{\mathbf{C}}{e^{A_{t_0}}} \int_{D_v} (v''_{t_0,\varepsilon} \circ \Psi) |F^{1+\delta}|^2 e^{-\varphi - \Psi} d\lambda_n. \end{aligned} \quad (5.29)$$

Given t_0 and D_v , it is clear that $(v''_{t_0,\varepsilon} \circ \Psi) |F^{1+\delta}|^2 e^{-\varphi - \Psi}$ have a uniform bound on D_v independent of ε . Then the integrals $\int_{D_v} (v''_{t_0,\varepsilon} \circ \Psi) |F^{1+\delta}|^2 e^{-\varphi - \Psi} d\lambda_n$ have a uniform bound independent of ε , for any given t_0 and D_v .

As $|(v'_{t_0,\varepsilon} \circ \Psi)F^{1+\delta}|^2 e^{-\varphi}$ have a uniform bound on D_v independent of ε , it follows that the integrals $\int_{D_v} |(1 - v'_{t_0,\varepsilon} \circ \Psi)F^{1+\delta}|^2 e^{-\varphi} d\lambda_n$ have a uniform bound independent of ε , for any given t_0 and D_v .

As

$$\begin{aligned} & \int_{D_v} |F_{v,t_0,\varepsilon}|^2 e^{-\varphi} d\lambda_n \\ & \leq \int_{D_v} |F_{v,t_0,\varepsilon} - (1 - v'_{t_0,\varepsilon} \circ \Psi)F^{1+\delta}|^2 e^{-\varphi} d\lambda_n + \int_{D_v} |(1 - v'_{t_0,\varepsilon} \circ \Psi)F^{1+\delta}|^2 e^{-\varphi} d\lambda_n \\ & \leq \frac{\mathbf{C}}{e^{A_{t_0}}} \int_{D_v} (v''_{t_0,\varepsilon} \circ \Psi) |F^{1+\delta}|^2 e^{-\varphi - \Psi} d\lambda_n + \int_{D_v} |(1 - v'_{t_0,\varepsilon} \circ \Psi)F^{1+\delta}|^2 e^{-\varphi} d\lambda_n, \end{aligned} \quad (5.30)$$

then the integrals $\int_{D_v} |F_{v,t_0,\varepsilon}|^2 e^{-\varphi} d\lambda_n$ have a uniform bound independent of ε .

As $\bar{\partial}F_{v,t_0,\varepsilon} = 0$ when $\varepsilon \rightarrow 0$ and the unit ball of $L^2_\varphi(D_v)$ is weakly compact, it follows that the weak limit of some weakly convergent subsequence of $\{F_{v,t_0,\varepsilon}\}_\varepsilon$ gives us a holomorphic function F_{v,t_0} on Δ^n .

Then we can also choose a subsequence of the weakly convergent subsequence of $\{F_{v,t_0,\varepsilon}\}_\varepsilon$, such that the chosen sequence is uniformly convergent on any compact subset of D_v , denoted by $\{F_{v,t_0,\varepsilon}\}_\varepsilon$ without ambiguity.

For any given compact subset K_0 on D_v , $F_{v,t_0,\varepsilon}$, $|(1 - v'_{t_0,\varepsilon} \circ \Psi)F^{1+\delta}|^2 e^{-\varphi}$ and $(v''_{t_0,\varepsilon} \circ \Psi)|F^{1+\delta}|^2 e^{-\varphi-\Psi}$ have uniform bounds on K_0 independent of ε .

By inequality 5.28, it follows that

$$(F_{v,t_0} - F^{1+\delta}, o) \in \mathcal{I}(\varphi + \Psi)_o.$$

Using the dominated convergence theorem on any compact subset K of D_v , we obtain

$$\begin{aligned} & \int_K |F_{v,t_0} - (1 - b_{t_0}(\Psi))F^{1+\delta}|^2 e^{-\varphi} d\lambda_n \\ & \leq (1 + \frac{1}{\delta}) \int_{D_v} \frac{1}{B_0} (\mathbb{I}_{\{-t_0 - B_0 < t < -t_0\}} \circ \Psi) |F^{1+\delta}|^2 e^{-\varphi-\Psi} d\lambda_n. \end{aligned} \quad (5.31)$$

Proposition 4.1 has thus been proved.

It suffices to find η and ϕ such that $(\eta + g^{-1}) \leq \mathbf{C}e^{-\Psi_m}e^{-\phi} = \mathbf{C}\tilde{\mu}^{-1}$ on D_v . As $\eta = s(-v_{t_0,\varepsilon} \circ \Psi_m)$ and $\phi = u(-v_{t_0,\varepsilon} \circ \Psi_m)$, we have $(\eta + g^{-1})e^{v_{t_0,\varepsilon} \circ \Psi_m}e^{\phi} = (s + \frac{s'^2}{u''s - s''})e^{-t}e^u \circ (-v_{t_0,\varepsilon} \circ \Psi_m)$.

Summarizing the above discussion about s and u , we are naturally led to a system of ODEs:

$$\begin{aligned} 1). & (s + \frac{s'^2}{u''s - s''})e^{u-t} = \mathbf{C}, \\ 2). & s' - su' = 1, \end{aligned} \quad (5.32)$$

where $t \in [0, +\infty)$, and $\mathbf{C} = 1$.

It is not hard to solve the ODE system 5.32 (using the same arguments in [16] [18] and noting the boundary condition) and get $u = -\log(1 + \frac{1}{\delta} - e^{-t})$ and $s = \frac{(1+\frac{1}{\delta})t + \frac{1}{\delta}(1+\frac{1}{\delta})}{1+\frac{1}{\delta}-e^{-t}} - 1$. It follows that $s \in C^\infty((0, +\infty))$ satisfies $s \geq \frac{1}{\delta}$, $u' \leq 0$ and $u \in C^\infty((0, +\infty))$ satisfies $u''s - s'' > 0$.

As $u = -\ln(1 + \frac{1}{\delta} - e^{-t})$ is decreasing with respect to t , then

$$\frac{\mathbf{C}}{e^{A_{t_0}}} = \frac{1}{\exp \inf_{t \geq t_0} u(t)} = \sup_{t \geq t_0} \frac{1}{e^{u(t)}} = \sup_{t \geq t_0} (1 + \frac{1}{\delta} - e^{-t}) = 1 + \frac{1}{\delta},$$

for any $t_0 \geq 0$, therefore we are done.

6. DISCUSSION OF INEQUALITY 1.3

We would like to give a proof of and a Remark on inequality 1.3.

6.1. Proof of inequality 1.3.

It suffices to prove

$$\frac{t}{6(t-1)} < (\frac{1}{(t-1)(2t-1)})^{\frac{1}{t}}. \quad (6.1)$$

We consider the function

$$P(t) = \frac{1}{t} \log \frac{1}{(t-1)(2t-1)} + \log \frac{t-1}{t}.$$

Replacing t by $\frac{1}{x}$, we obtain

$$\begin{aligned} Q(x) &= P\left(\frac{1}{x}\right) = \frac{1}{\frac{1}{x}} \log \frac{1}{\left(\frac{1}{x} - 1\right)\left(2\frac{1}{x} - 1\right)} + \log \frac{\frac{1}{x} - 1}{\frac{1}{x}} \\ &= \log\left(\left(\frac{x^2}{(1-x)(2-x)}\right)^x (1-x)\right) \\ &= 2x \log x + (1-x) \log(1-x) - x \log(2-x) \end{aligned} \quad (6.2)$$

We need to prove that inequality $e^{Q(x)} > \frac{1}{6}$ holds for any $x \in (0, 1)$. One can obtain the derivative $Q'(x)$ of $Q(x)$ as follows

$$\begin{aligned} Q'(x) &= (2x \log x + (1-x) \log(1-x) - x \log(2-x))' \\ &= (2 \log x + 2) + (-1 - \log(1-x)) + \left(\frac{x}{2-x} - \log(2-x)\right) \\ &= 2 \log x - \log(1-x) + \frac{2}{2-x} - \log(2-x) \end{aligned} \quad (6.3)$$

One can obtain the derivative $Q''(x)$ of $Q(x)$ as follows

$$\begin{aligned} Q''(x) &= (2 \log x - \log(1-x) + \frac{2}{2-x} - \log(2-x))' \\ &= \frac{2}{x} + \frac{1}{1-x} + \frac{2}{(2-x)^2} + \frac{1}{2-x} > 0 \end{aligned} \quad (6.4)$$

Note that

$$\lim_{x \rightarrow 0} Q(x) = 0,$$

$$\lim_{x \rightarrow 1} Q(x) = 0,$$

and $Q'(x)(\frac{1}{2}) > 0$, then the preimage of the minimal value must exist in $(0, \frac{1}{2})$.

Now we consider the minimal of the following three functions:

$$Q_1(x) := 2x \log x, \quad Q_2(x) := (1-x) \log(1-x), \quad Q_3(x) := -x \log(2-x).$$

It is clear that

$$Q(x) = Q_1(x) + Q_2(x) + Q_3(x),$$

and

$$\min_{x \in (0, \frac{1}{2})} Q(x) \geq \min_{x \in (0, \frac{1}{2})} Q_1(x) + \min_{x \in (0, \frac{1}{2})} Q_2(x) + \min_{x \in (0, \frac{1}{2})} Q_3(x).$$

It is known that $\min_{x \in (0, \frac{1}{2})} Q_1(x) = -\frac{2}{e}$, $\min_{x \in (0, \frac{1}{2})} Q_2(x) > -\log \sqrt{2}$, and $\min_{x \in (0, \frac{1}{2})} Q_3(x) = Q_3(\frac{1}{2}) > -\log \sqrt{\frac{3}{2}}$. Then we obtain that

$$\min_{x \in (0, 1)} Q(x) \geq \min_{x \in (0, \frac{1}{2})} Q(x) > -\frac{2}{e} - \log \sqrt{2} - \log \sqrt{\frac{3}{2}},$$

that is to say

$$\min_{x \in (0, 1)} e^{Q(x)} > \frac{1}{\sqrt{3}e^{\frac{2}{e}}} > \frac{1}{6}. \quad (6.5)$$

Thus we obtain inequality 1.3.

6.2. A Remark on inequality 1.3.

In this subsection, we give a remark about the accuracy our effectiveness of Corollary 1.4.

When $D = \Delta$, and $z_0 = 0$, $\varphi = \frac{1}{p} \log |z|^2$, it is clear that $\|1\|_{\varphi}^2 K(z_0) = \frac{1}{1 - \frac{1}{p}}$.

Using Corollary 1.4 and inequality 6.5, we obtain

$$\|1\|_{\varphi}^2 K(z_0) > \frac{1}{\sqrt{3}e^{\frac{2}{e}}} \frac{1}{1 - \frac{1}{p}}$$

for any D and any $z_0 \in D$.

One can obtain that $\min_{x \in (0,1)} e^{Q(x)} > 0.2876$, which gives a more precise form of inequality 1.3:

$$\frac{1}{400(t-1)} < 0.2876 \frac{t}{(t-1)} < \left(\frac{1}{(t-1)(2t-1)} \right)^{\frac{1}{t}}. \quad (6.6)$$

Then we obtain

$$\|1\|_{\varphi}^2 K(z_0) > 0.2876 \frac{1}{1 - \frac{1}{p}}$$

for any D and any $z_0 \in D$.

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